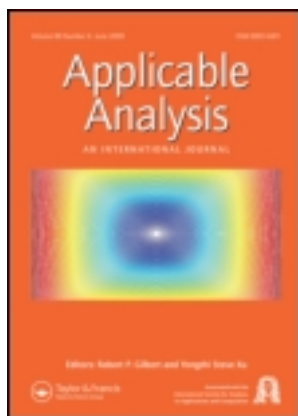


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Blow-up of solutions to systems of nonlinear wave equations with supercritical sources†

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In this article, we focus on the life span of solutions to the following system of nonlinear wave equations:

$$\begin{aligned}u_{tt} - \Delta u + g_1(u_t) &= f_1(u, v) \\v_{tt} - \Delta v + g_2(v_t) &= f_2(u, v)\end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with Robin and Dirichlet boundary conditions on u and v , respectively. The nonlinearities $f_1(u, v)$ and $f_2(u, v)$ represent strong sources of supercritical order, while $g_1(u_t)$ and $g_2(v_t)$ represent interior damping. The nonlinear boundary condition on u , namely $\partial_\nu u + u + g(u_t) = h(u)$ on Γ , also features $h(u)$, a boundary source, and $g(u_t)$, a boundary damping. Under some restrictions on the parameters, we prove that every weak solution to system above blows up in finite time, provided the initial energy is negative.

Keywords: blow-up; nonlinear wave equations; damping and source terms; weak solutions; energy identity

AMS Subject Classifications: Primary 35L05, 35L20; Secondary 58J45

1. Introduction

1.1. Preliminaries

Wave equations under the influence of nonlinear damping and nonlinear sources have generated considerable interest over recent years. As the linear theory has been substantially developed, many problems for systems with supercritical nonlinearities remain open. In this article, we study a system of coupled nonlinear wave equations which features two competing forces, one force is damping and the other is a strong source. Our main interest here is to investigate the possibility of the finite time blow-up of solutions, under nominal conditions.

For the sake of clarity, we restrict our analysis to the physically more relevant case when $\Omega \subset \mathbb{R}^3$. Our results easily extend to bounded domains in \mathbb{R}^n , by

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†Dedicated to the memory of Alan Jeffrey.

accounting for the corresponding Sobolev embeddings, and accordingly adjusting the conditions imposed on the parameters. Thus, throughout this article we assume that Ω is bounded, open and connected non-empty set in \mathbb{R}^3 with a smooth boundary $\Gamma = \partial\Omega$.

We study the following system of nonlinear wave equations:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f_1(u, v), & \text{in } \Omega \times (0, T), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u, v), & \text{in } \Omega \times (0, T), \\ \partial_\nu u + u + g(u_t) = h(u), & \text{on } \Gamma \times (0, T), \\ v = 0, & \text{on } \Gamma \times (0, T), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H_0^1(\Omega), v_t(0) = v_1 \in L^2(\Omega), \end{cases} \quad (1.1)$$

where the nonlinearities $f_1(u, v)$, $f_2(u, v)$ and $h(u)$ represent interior and boundary sources, while $g_1(u_t)$, $g_2(v_t)$ and $g(u_t)$ act as interior and boundary damping. The source–damping interaction in (1.1) encompasses a broad class of problems in quantum field theory and certain mechanical applications [1,2], whereas *non-dissipative* ‘energy-building’ sources, especially those on the boundary, arise whenever one considers a wave equation being coupled with other types of dynamics, such as structure–acoustic or fluid–structure interaction models [3]. In light of these applications we are mainly interested in higher order nonlinearities, as described in the following assumption.

ASSUMPTION 1.1

- **Interior sources:** $f_j(u, v) \in C^1(\mathbb{R}^2)$ such that

$$|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1), \quad j = 1, 2, \quad \text{with } 1 \leq p < 6.$$

- **Boundary source:** $h \in C^1(\mathbb{R})$ such that

$$|h'(s)| \leq C(|s|^{k-1} + 1), \quad \text{with } 1 \leq k < 4.$$

- **Damping:** g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0) = g_2(0) = g(0) = 0$. In addition, the following growth conditions hold: there exist positive constants a_j and $b_j, j = 1, 2, 3$, such that, for all $s \in \mathbb{R}$,

$$a_1 |s|^{m+1} \leq g_1(s)s \leq b_1 |s|^{m+1}, \quad \text{with } m \geq 1,$$

$$a_2 |s|^{r+1} \leq g_2(s)s \leq b_2 |s|^{r+1}, \quad \text{with } r \geq 1,$$

$$a_3 |s|^{q+1} \leq g(s)s \leq b_3 |s|^{q+1}, \quad \text{with } q \geq 1.$$

- **Parameters:** $\max\{p \frac{m+1}{m}, p \frac{r+1}{r}\} < 6, k \frac{q+1}{q} < 4$.

Let us note here that if the damping terms $g_1(u_t)$, $g_2(v_t)$ and $g(u_t)$ are removed from the system, then the presence of any of the source terms should drive the solution of (1.1) to blow-up in finite time. In such a case, one can appeal to a variety of methods (going back to the work of Glassey [4], Levine [5] and others) to show that *most* solutions to the problem blow up in finite time. In addition, if the source terms are removed from the system, then damping terms of various forms should yield existence of global solutions, (cf. [6–9]). However, when both damping and

source terms are present, especially on the boundary, the analysis of their interaction and their influence on the behaviour of solutions becomes more difficult (see, e.g. [10–15] and the references therein).

A well-known system, which is a special case of (1.1), is the following polynomially damped system studied extensively in the literature [16–19]:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u, v), & \text{in } \Omega \times (0, \infty), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u, v), & \text{in } \Omega \times (0, \infty), \end{cases} \quad (1.2)$$

where $f_1(u, v) = \partial_u F(u, v)$ and $f_2(u, v) = \partial_v F(u, v)$, and $F \in C^1(\mathbb{R}^2)$ is given by

$$F(u, v) = a|u + v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}, \quad (1.3)$$

where $p \geq 3$, $a > 1$ and $b > 0$.

It is worth noting here that systems of nonlinear wave equations such as (1.2) go back to Reed [20] who proposed a similar system in three space dimensions but without the presence of damping. Indeed, recently in [16] and later in [17] the authors studied system (1.2) with Dirichlét boundary conditions on both u and v where the exponent of the source was restricted to be *critical* ($p = 3$ in 3D). The more general system (1.1) with Robin boundary condition has been studied recently in [21], where the source terms are allowed to be of super-supercritical order (i.e. $1 \leq p < 6$, $1 \leq k < 4$). Indeed, the authors in [21] used monotone operator theory and nonlinear semigroups to obtain several results on local and global existence and uniqueness of weak solutions. The main goal of this article is to complement the work of [21] by establishing two blow-up theorems for (1.1).

Our results are inspired by the work of [11–13] for their treatment of a single wave equation. Although the basic calculus in the proofs of Theorem 1.8 and Theorem 1.9 draw from ideas in [12,13,16] and also from the recent results in [11], our proofs had to be significantly adjusted to accommodate the coupling in the system (1.1). For other relevant results on wave equations with source–damping interplay see [22–25] and the references therein.

It is important to note that the mixture of Robin and Dirichlét boundary conditions in the system (1.1) is neither essential to the methods used in this article nor to our results. Indeed, similar existence, uniqueness and blow-up results can be easily obtained if instead one imposes Robin boundary conditions on both u and v .

To this end, we point out that the following notations will be used throughout this article:

$$\begin{aligned} \|u\|_s &= \|u\|_{L^s(\Omega)}, & |u|_s &= \|u\|_{L^s(\Gamma)}, & \|u\|_{1,\Omega} &= \|u\|_{H^1(\Omega)}; \\ (u, v)_\Omega &= (u, v)_{L^2(\Omega)}, & (u, v)_\Gamma &= (u, v)_{L^2(\Gamma)}, & (u, v)_{1,\Omega} &= (u, v)_{H^1(\Omega)}. \end{aligned}$$

We also use the notation γu to denote the *trace* of u on Γ and we write $\frac{d}{dt}(\gamma u(t))$ as γu_t . We finally note that $(\|\nabla u\|_2^2 + |\gamma u|_2^2)^{1/2}$ is a norm equivalent to the standard $H^1(\Omega)$ -norm. This fact follows from a Poincaré–Wirtinger type of inequality:

$$\|u\|_2^2 \leq C(\|\nabla u\|_2^2 + |\gamma u|_2^2), \quad \text{for all } u \in H^1(\Omega).$$

Thus, throughout this article we put,

$$\|u\|_{1,\Omega}^2 = \|\nabla u\|_2^2 + |\gamma u|_2^2 \quad \text{and} \quad (u, v)_{1,\Omega} = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma,$$

for $u, v \in H^1(\Omega)$.

1.2. Existence theory

We begin by introducing the definition of a weak solution to (1.1).

Definition 1.2 A pair of functions (u, v) is said to be a *weak solution* of (1.1) on $[0, T]$ if

- $u \in C([0, T]; H^1(\Omega)), v \in C([0, T]; H_0^1(\Omega)), u_t \in C([0, T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0, T)), \gamma u_t \in L^{q+1}(\Gamma \times (0, T)), v_t \in C([0, T]; L^2(\Omega)) \cap L^{r+1}(\Omega \times (0, T));$
- $(u(0), v(0)) = (u_0, v_0) \in H^1(\Omega) \times H_0^1(\Omega), (u_t(0), v_t(0)) = (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega);$
- for all $t \in [0, T], u$ and v verify the following identities:

$$\begin{aligned} & (u_t(t), \phi(t))_\Omega - (u_t(0), \phi(0))_\Omega + \int_0^t [-(u_t(\tau), \phi_t(\tau))_\Omega + (u(\tau), \phi(\tau))_{1,\Omega}] d\tau \\ & + \int_0^t \int_\Omega g_1(u_t(\tau)) \phi(\tau) dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t(\tau)) \gamma \phi(\tau) d\Gamma d\tau \\ & = \int_0^t \int_\Omega f_1(u(\tau), v(\tau)) \phi(\tau) dx d\tau + \int_0^t \int_\Gamma h(\gamma u(\tau)) \gamma \phi(\tau) d\Gamma d\tau, \end{aligned} \tag{1.4}$$

$$\begin{aligned} & (v_t(t), \psi(t))_\Omega - (v_t(0), \psi(0))_\Omega + \int_0^t [-(v_t(\tau), \psi_t(\tau))_\Omega + (v(\tau), \psi(\tau))_{1,\Omega}] d\tau \\ & + \int_0^t \int_\Omega g_2(v_t(\tau)) \psi(\tau) dx d\tau = \int_0^t \int_\Omega f_2(u(\tau), v(\tau)) \psi(\tau) dx d\tau, \end{aligned} \tag{1.5}$$

for all test functions satisfying: $\phi \in C([0, T]; H^1(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$ such that $\gamma \phi \in L^{q+1}(\Gamma \times (0, T))$ with $\phi_t \in L^1([0, T]; L^2(\Omega))$ and $\psi \in C([0, T]; H_0^1(\Omega)) \cap L^{r+1}(\Omega \times (0, T))$ such that $\psi_t \in L^1([0, T]; L^2(\Omega))$.

In order to state our main results, it is essential to make a connection with the recent results in [21]. Thus, for the reader's convenience, we summarize some of the main results in [21] in the following theorem.

THEOREM 1.3 (Local and global weak solutions [21]) *Assume the validity of Assumption 1.1, then there exists a local weak solution (u, v) to (1.1) defined on $[0, T]$, for some $T > 0$. Moreover, (u, v) satisfies the following energy identity for all $t \in [0, T]$:*

$$\begin{aligned} & \mathcal{E}(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t] dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t d\Gamma d\tau \\ & = \mathcal{E}(0) + \int_0^t \int_\Omega [f_1(u, v)u_t + f_2(u, v)v_t] dx d\tau + \int_0^t \int_\Gamma h(\gamma u) \gamma u_t d\Gamma d\tau, \end{aligned} \tag{1.6}$$

where the quadratic energy is given by

$$\mathcal{E}(t) = \frac{1}{2} (\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2). \tag{1.7}$$

If, in addition, we assume $p \leq \min\{m, r\}, k \leq q$ and $u_0, v_0 \in L^{p+1}(\Omega), \gamma u_0 \in L^{k+1}(\Gamma)$, then the said solution (u, v) is a global weak solution and T can be taken arbitrarily large.

In order to state the uniqueness results in [21], we shall need additional assumptions on the sources and the boundary damping.

ASSUMPTION 1.4

- (a) For $p > 3$, there exists a function $F(u, v) \in C^3(\mathbb{R}^2)$ such that $f_1(u, v) = \partial_u F(u, v)$, $f_2(u, v) = \partial_v F(u, v)$, and $|D^\alpha F(u, v)| \leq C(|u|^{p-2} + |v|^{p-2} + 1)$ for all multi-indices $|\alpha| = 3$ and all $u, v \in \mathbb{R}$.
- (b) For $k \geq 2$, $h \in C^2(\mathbb{R})$ such that $|h'(s)| \leq C(|s|^{k-2} + 1)$, for all $s \in \mathbb{R}$.
- (c) For $k < 2$, there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g |s_1 - s_2|^2$, for all $s_1, s_2 \in \mathbb{R}$.

The following uniqueness results are based on the validity of Assumptions 1.1 and 1.4. However, in the case when the interior sources f_1 and f_2 fail to satisfy Assumption 1.4(a), as in system (1.2) for the values $3 < p \leq 5$; we still can prove uniqueness of solutions of (1.1), provided the exponents m and r of the interior damping are sufficiently large.

THEOREM 1.5 (Uniqueness of weak solutions [21]) *Assume that one of the following statements holds:*

- Assumptions 1.1 and 1.4 are valid, $u_0, v_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$ and $\gamma u_0 \in L^{2(k-1)}(\Gamma)$.
- Assumption 1.1 and Assumptions 1.4(b), (c) are valid, $u_0, v_0 \in L^{3(p-1)}(\Omega)$, $\gamma u_0 \in L^{2(k-1)}(\Gamma)$, and $m, r \geq 3p - 4$, if $p > 3$.

Then, weak solutions of (1.1) are unique.

1.3. Main results

In order to state our blow-up results, we need additional assumptions on interior and boundary sources and initial data.

ASSUMPTION 1.6

- There exists a function $F \in C^2(\mathbb{R}^2)$ such that $f_1(u, v) = \partial_u F(u, v)$ and $f_2(u, v) = \partial_v F(u, v)$, $(u, v) \in \mathbb{R}^2$. Moreover, there exist $c_0 > 0$ and $c_1 > 2$ such that $F(u, v) \geq c_0(|u|^{p+1} + |v|^{p+1})$ and $uf_1(u, v) + vf_2(u, v) \geq c_1 F(u, v)$, for all $(u, v) \in \mathbb{R}^2$.
- There exist $c_2 > 0$ and $c_3 > 2$ such that $H(s) \geq c_2 |s|^{k+1}$ and $h(s)s \geq c_3 H(s)$, for all $s \in \mathbb{R}$, where $H(s) = \int_0^s h(\tau) d\tau$.
- The initial energy $E(0) < 0$, where the total energy $E(t)$ is given by

$$E(t) = \frac{1}{2} (\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) - \int_{\Omega} F(u(t), v(t)) dx - \int_{\Gamma} H(\gamma u(t)) d\Gamma. \tag{1.8}$$

Remark 1.7 It is important to note here that our restrictions on interior and boundary sources in Assumption 1.6 are natural and quite reasonable. For instance, the function F given in (1.3) satisfies Assumption 1.6. Indeed, a quick calculations show that there exists a constant $c_0 > 0$ such that $F(u, v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, provided b is chosen large enough. Moreover, it is easy to compute and find that $uf_1(u, v) + vf_2(u, v) = (p + 1)F(u, v)$. Since the blow-up theorems below require $p > m \geq 1$, then $p + 1 > 2$, and so, the assumption $c_1 > 2$ is reasonable. A simple

example of a boundary source term that satisfies Assumption 1.6 is $h(s) = |s|^{k-1}s$. In this case, $H(s) = \frac{1}{k+1}|s|^{k+1}$, and so, $h(s)s = (k+1)H(s)$. Again, the statement of Theorem 1.8 requires $k > q \geq 1$, implies that $k+1 > 2$. Thus, the restriction $c_3 > 2$ in Assumption 1.6 is also reasonable.

Our first blow-up result shows that if the interior and boundary sources are more dominant than their corresponding damping terms, and the initial energy is negative, then every weak solution of (1.1) blows up in finite time. In addition, we obtain an upper bound for the life span of solutions.

THEOREM 1.8 (Blow-up of solutions – Part 1) *Assume the validity of Assumptions 1.1 and 1.6. If $p > \max\{m, r\}$ and $k > q$, then any weak solution (u, v) of (1.1) blows up in finite time. More precisely, $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \rightarrow \infty$ as $t \rightarrow T^-$, for some $0 < T < \infty$.*

Our second result shows that all solutions of (1.1) blows up in finite time, provided $E(0) < 0$, and the interior sources dominate both interior and boundary damping, without any restriction on the boundary source.

THEOREM 1.9 (Blow-up of solutions – Part 2) *Assume the validity of Assumptions 1.1 and 1.6. If $p > \max\{m, r, 2q - 1\}$, then any weak solution (u, v) of (1.1) blows up in finite time. Specifically, $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \rightarrow \infty$ as $t \rightarrow T^-$, for some $0 < T < \infty$.*

Remark 1.10 Although the existence and uniqueness results in Theorems 1.3 and 1.5 hold for sources that are super-supercritical (i.e. $p < 6$ and $k < 4$), however the assumptions in Theorems 1.8 and 1.9 force the restrictions $p < 5$ and $k < 3$. To see this, we note that both theorems require $p > m$, and by Assumption 1.1, it follows that, $6 > p(1 + \frac{1}{m}) > p(1 + \frac{1}{p}) = p + 1$, which implies $p < 5$. By the same observation, we conclude $k < 3$ in Theorem 1.8. Although $k > q$ is not required by Theorem 1.9, we still must have $k < 3$. Indeed, since $2q - 1 < p < 5$, then $q < 3$. Whence, by Assumption 1.1, we have $4 > k(1 + \frac{1}{q}) > \frac{4}{3}k$, and so, $k < 3$.

2. Proof of Theorem 1.8

Proof Let (u, v) be a weak solution to (1.1) in the sense of Definition 1.2. Throughout the proof, we assume the validity of Assumptions 1.1 and 1.6, $p > \max\{m, r\}$ and $k > q$. We define the life span T of such a solution (u, v) to be the supremum of all $T^* > 0$ such that (u, v) is a solution to (1.1) in the sense of Definition (1.2) on $[0, T^*]$. Our goal is to show that T is necessarily finite, and obtain an upper bound for T .

As in [11,16], for $t \in [0, T)$, we define:

$$\begin{aligned} G(t) &= -E(t), \\ N(t) &= \|u(t)\|_2^2 + \|v(t)\|_2^2, \\ S(t) &= \int_{\Omega} F(u(t), v(t))dx + \int_{\Gamma} H(\gamma u(t))d\Gamma. \end{aligned}$$

It follows that,

$$G(t) = -\frac{1}{2} (\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) + S(t), \tag{2.1}$$

and

$$N'(t) = 2 \int_{\Omega} [u(t)u_t(t) + v(t)v_t(t)]dx. \tag{2.2}$$

Moreover, by the assumptions $H(s) \geq c_2|s|^{k+1}$ and $F(u, v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, one has

$$S(t) \geq c_0 \left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} \right) + c_2 |\gamma u(t)|_{k+1}^{k+1}. \tag{2.3}$$

Let

$$0 < \alpha < \min \left\{ \frac{1}{m+1} - \frac{1}{p+1}, \frac{1}{r+1} - \frac{1}{p+1}, \frac{1}{q+1} - \frac{1}{k+1}, \frac{p-1}{2(p+1)} \right\}. \tag{2.4}$$

In particular, $\alpha < \frac{1}{2}$. To simplify the notation, we introduce the following constants:

$$\begin{aligned} K_1 &= b_1 |\Omega|^{\frac{p-m}{(p+1)(m+1)}} c_0^{-\frac{1}{p+1}}, & K_2 &= b_2 |\Omega|^{\frac{p-r}{(p+1)(r+1)}} c_0^{-\frac{1}{p+1}}, & K_3 &= b_3 |\Gamma|^{\frac{k-q}{(k+1)(q+1)}} c_2^{-\frac{1}{k+1}}, \\ \delta_1 &= \frac{\lambda}{6} G(0)^{\frac{1}{m+1} - \frac{1}{p+1}}, & \delta_2 &= \frac{\lambda}{6} G(0)^{\frac{1}{r+1} - \frac{1}{p+1}}, & \delta_3 &= \frac{\lambda}{6} G(0)^{\frac{1}{q+1} - \frac{1}{k+1}}, \end{aligned} \tag{2.5}$$

where $\lambda = \min\{c_1 - 2, c_3 - 2\} > 0$, and $|\Omega|$, $|\Gamma|$ denote the Lebesgue measures of Ω and Γ .

Note that the energy identity (1.6) is equivalent to

$$G(t) = G(0) + \int_0^t \int_{\Omega} [g_1(u_t)u_t + g_2(v_t)v_t]dx \, d\tau + \int_0^t \int_{\Gamma} g(\gamma u_t)\gamma u_t \, d\Gamma \, d\tau.$$

So, by Assumption 1.1 and the regularity of the solution (u, v) , we conclude that $G(t)$ is absolutely continuous and

$$\begin{aligned} G'(t) &= \int_{\Omega} [g_1(u_t(t))u_t(t) + g_2(v_t(t))v_t(t)]dx + \int_{\Gamma} g(\gamma u_t(t))\gamma u_t(t)d\Gamma \\ &\geq a_1 \|u_t(t)\|_{m+1}^{m+1} + a_2 \|v_t(t)\|_{r+1}^{r+1} + a_3 |\gamma u_t(t)|_{q+1}^{q+1} \geq 0, \quad \text{a.e. } [0, T). \end{aligned} \tag{2.6}$$

Thus, $G(t)$ is non-decreasing. Since $G(0) = -E(0) > 0$, then it follows that

$$0 < G(0) \leq G(t) \leq S(t) \quad \text{for } 0 \leq t < T. \tag{2.7}$$

Now, put

$$Y(t) = G(t)^{1-\alpha} + \epsilon N'(t), \tag{2.8}$$

where $0 < \epsilon \leq G(0)$. Later in the proof we further adjust the requirements on ϵ . We shall show that

$$Y'(t) = (1 - \alpha)G(t)^{-\alpha}G'(t) + \epsilon N''(t), \tag{2.9}$$

where

$$\begin{aligned} N''(t) &= 2 \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \right) - 2 \left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \right) \\ &\quad - 2 \int_{\Omega} (g_1(u_t)u + g_2(v_t)v)dx - 2 \int_{\Gamma} g(\gamma u_t)\gamma u \, d\Gamma \\ &\quad + 2 \int_{\Omega} (f_1(u, v)u + f_2(u, v)v)dx + 2 \int_{\Gamma} h(\gamma u)\gamma u \, d\Gamma, \quad \text{a.e. } [0, T). \end{aligned} \tag{2.10}$$

In order to obtain (2.10), we first verify $u \in L^{m+1}(\Omega \times (0, t))$ for all $t \in [0, T)$. Indeed, since both u and $u_t \in C([0, t]; L^2(\Omega))$, we can write

$$\begin{aligned} \int_0^t \int_{\Omega} |u|^{m+1} dx d\tau &= \int_0^t \int_{\Omega} \left| \int_0^{\tau} u_t(s) ds + u_0 \right|^{m+1} dx d\tau \\ &\leq 2^m \left[\int_0^t \int_{\Omega} \left| \int_0^{\tau} u_t(s) ds \right|^{m+1} dx d\tau + \int_0^t \int_{\Omega} |u_0|^{m+1} dx d\tau \right] \\ &\leq 2^m \left[t^m \int_0^t \int_{\Omega} \int_0^t |u_t(s)|^{m+1} ds dx d\tau + t \|u_0\|_{m+1}^{m+1} \right] \\ &\leq 2^m \left(t^{m+1} \|u_t\|_{L^{m+1}(\Omega \times (0, t))}^{m+1} + t \|u_0\|_{m+1}^{m+1} \right) < \infty, \end{aligned} \tag{2.11}$$

for all $t \in [0, T)$, where we have used the regularity enjoyed by u , namely the fact $u_t \in L^{m+1}(\Omega \times (0, t))$, and the assumption $u_0 \in H^1(\Omega) \hookrightarrow L^{m+1}(\Omega)$ since $m < p < 5$, as stated in Remark 1.10. Hence, $u \in L^{m+1}(\Omega \times (0, t))$ for all $t \in [0, T)$. Likewise, one can show that $v \in L^{r+1}(\Omega \times (0, t))$ for all $t \in [0, T)$. Moreover, by similar estimates as in (2.11), we deduce

$$\|\gamma u\|_{L^{q+1}(\Gamma \times (0, t))}^{q+1} \leq 2^q \left(t^{q+1} \|\gamma u_t\|_{L^{q+1}(\Gamma \times (0, t))}^{q+1} + t \|\gamma u_0\|_{q+1}^{q+1} \right) < \infty.$$

Thus, $\gamma u \in L^{q+1}(\Gamma \times (0, t))$, for all $t \in [0, T)$.

The above shows that u and v enjoy, respectively, the regularity restrictions imposed on the test functions ϕ and ψ , as stated in Definition 1.2. Therefore, we can replace ϕ by u in (1.4) and ψ by v in (1.5), and by (2.2), we obtain

$$\begin{aligned} \frac{1}{2} N'(t) &= \int_{\Omega} (u_1 u_0 + v_1 v_0) dx + \int_0^t \int_{\Omega} (|u_t|^2 + |v_t|^2) dx d\tau - \int_0^t (\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) d\tau \\ &\quad - \int_0^t \int_{\Omega} (g_1(u_t)u + g_2(v_t)v) dx d\tau - \int_0^t \int_{\Gamma} g(\gamma u_t) \gamma u d\Gamma d\tau \\ &\quad + \int_0^t \int_{\Omega} (f_1(u, v)u + f_2(u, v)v) dx d\tau + \int_0^t \int_{\Gamma} h(\gamma u) \gamma u d\Gamma d\tau, \quad \text{a.e. } [0, T). \end{aligned} \tag{2.12}$$

By Assumption 1.1, $|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, and so, by the mean value theorem, one has $|f_j(u, v)| \leq C(|u|^p + |v|^p + 1)$, $j = 1, 2$. Thus, by using Young and Hölder's inequality, we have

$$\begin{aligned} \int_0^t \left| \int_{\Omega} (f_1(u, v)u + f_2(u, v)v) dx \right| d\tau &\leq C \int_0^t \int_{\Omega} (|u|^p + |v|^p + 1)(|u| + |v|) dx d\tau \\ &\leq C_T \int_0^t \int_{\Omega} (|u|^{p+1} + |v|^{p+1}) dx dt < \infty, \end{aligned} \tag{2.13}$$

for all $t \in [0, T)$, where we have used the fact $u \in C([0, t]; H^1(\Omega))$, the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the restriction $p < 5$, as mentioned in Remark 1.10.

In addition, by using the regularity of the solution (u, v) and the assumptions on the parameters, we infer

$$\int_0^t \left| \int_{\Omega} (g_1(u_t)u + g_2(v_t)v) dx \right| d\tau + \int_0^t \left| \int_{\Gamma} g(\gamma u_t) \gamma u d\Gamma \right| d\tau + \int_0^t \left| \int_{\Gamma} h(\gamma u) \gamma u d\Gamma \right| d\tau < \infty, \tag{2.14}$$

for all $t \in [0, T)$. Hence, it follows from (2.12)–(2.14), and the regularity of (u, v) that $N'(t)$ is absolutely continuous, and thus (2.10) follows.

Now, note that (2.1) yields

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 = -(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2) + 2S(t) - 2G(t). \tag{2.15}$$

So, by (2.9), (2.10), (2.15) and the assumptions $uf_1(u, v) + vf_2(u, v) \geq c_1F(u, v)$, $h(s)s \geq c_3H(s)$, we deduce

$$\begin{aligned} Y'(t) &\geq (1 - \alpha)G(t)^{-\alpha}G'(t) + 4\epsilon\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) + 4\epsilon G(t) \\ &\quad + 2\epsilon(c_1 - 2) \int_{\Omega} F(u(t), v(t))dx + 2\epsilon(c_3 - 2) \int_{\Gamma} H(\gamma u(t))d\Gamma \\ &\quad - 2\epsilon \int_{\Omega} g_1(u_t(t))u(t)dx - 2\epsilon \int_{\Omega} g_2(v_t(t))v(t)dx - 2\epsilon \int_{\Gamma} g(\gamma u_t(t))\gamma u(t)d\Gamma. \end{aligned} \tag{2.16}$$

We begin by estimating the last three terms on the right-hand side of (2.16). First, by using the assumption $g_1(s)s \leq b_1|s|^{m+1}$, Hölder's inequality, the fact $p > m$ and the inequality (2.3), we have

$$\begin{aligned} \left| \int_{\Omega} g_1(u_t(t))u(t)dx \right| &\leq b_1 \int_{\Omega} |u(t)||u_t(t)|^m dx \leq b_1 \|u(t)\|_{m+1} \|u_t(t)\|_{m+1}^m \\ &\leq b_1 |\Omega|^{\frac{p-m}{p+1(m+1)}} \|u(t)\|_{p+1} \|u_t(t)\|_{m+1}^m \leq K_1 S(t)^{\frac{1}{p+1}} \|u_t(t)\|_{m+1}^m \end{aligned} \tag{2.17}$$

where K_1 is defined in (2.5). Observe, the definition of α implies $\frac{1}{p+1} - \frac{1}{m+1} + \alpha < 0$. Therefore, by using (2.6)–(2.7), Young's inequality and recalling the definition of $\delta_1, \delta_2, \delta_3$ in (2.5), we obtain from (2.17) that

$$\begin{aligned} \left| \int_{\Omega} g_1(u_t(t))u(t)dx \right| &\leq K_1 S(t)^{\frac{1}{p+1} - \frac{1}{m+1}} S(t)^{\frac{1}{m+1}} \|u_t(t)\|_{m+1}^m \\ &\leq G(t)^{\frac{1}{p+1} - \frac{1}{m+1}} \left(\delta_1 S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}} \|u_t(t)\|_{m+1}^{m+1} \right) \\ &\leq \delta_1 G(t)^{\frac{1}{p+1} - \frac{1}{m+1}} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}} a_1^{-1} G'(t)G(t)^{-\alpha} G(t)^{\frac{1}{p+1} - \frac{1}{m+1} + \alpha} \\ &\leq \delta_1 G(0)^{\frac{1}{p+1} - \frac{1}{m+1}} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}} a_1^{-1} G'(t)G(t)^{-\alpha} G(0)^{\frac{1}{p+1} - \frac{1}{m+1} + \alpha}. \end{aligned} \tag{2.18}$$

By repeating the estimates (2.17)–(2.18), replacing $u(t)$ by $v(t)$ and m by r , we deduce

$$\left| \int_{\Omega} g_2(v_t(t))v(t)dx \right| \leq \delta_2 G(0)^{\frac{1}{p+1} - \frac{1}{r+1}} S(t) + C_{\delta_2} K_2^{\frac{r+1}{r}} a_2^{-1} G'(t)G(t)^{-\alpha} G(0)^{\frac{1}{p+1} - \frac{1}{r+1} + \alpha}. \tag{2.19}$$

Likewise, by replacing $u(t)$ by $\gamma u(t)$, Ω by Γ , p by k , m by q in (2.17)–(2.18), we obtain

$$\left| \int_{\Gamma} g(\gamma u_t(t))\gamma u(t)d\Gamma \right| \leq \delta_3 G(0)^{\frac{1}{k+1} - \frac{1}{q+1}} S(t) + C_{\delta_3} K_3^{\frac{q+1}{q}} a_3^{-1} G'(t)G(t)^{-\alpha} G(0)^{\frac{1}{k+1} - \frac{1}{q+1} + \alpha}. \tag{2.20}$$

Now, since $0 < \alpha < \frac{1}{2}$, we may choose $0 < \epsilon < 1$ small enough such that

$$\begin{aligned} L := 1 - \alpha - 2\epsilon \left(C_{\delta_1} K_1^{\frac{m+1}{m}} a_1^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{m+1} + \alpha} \right. \\ \left. + C_{\delta_2} K_2^{\frac{r+1}{r}} a_2^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{r+1} + \alpha} + C_{\delta_3} K_3^{\frac{q+1}{q}} a_3^{-1} G(0)^{\frac{1}{k+1} - \frac{1}{q+1} + \alpha} \right) \geq 0. \end{aligned} \tag{2.21}$$

In addition, since $\lambda = \min\{c_1 - 2, c_3 - 2\}$, then

$$(c_1 - 2) \int_{\Omega} F(u(t), v(t)) dx + (c_3 - 2) \int_{\Gamma} H(\gamma u(t)) d\Gamma \geq \lambda S(t). \tag{2.22}$$

Hence, by inserting (2.18)–(2.20) into (2.16) and using (2.21), (2.22) and (2.5), we conclude

$$\begin{aligned} Y'(t) &\geq LG(t)^{-\alpha} G'(t) + 4\epsilon \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \right) + 4\epsilon G(t) + \lambda \epsilon S(t) \\ &\geq 4\epsilon \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + G(t) \right) + \lambda \epsilon S(t). \end{aligned} \tag{2.23}$$

In particular, the inequality (2.23) shows that $Y(t)$ is increasing on $[0, T)$, with

$$Y(t) = G(t)^{1-\alpha} + \epsilon N'(t) \geq G(0)^{1-\alpha} + \epsilon N'(0). \tag{2.24}$$

If $N'(0) \geq 0$, then no further condition on ϵ is needed. However, if $N'(0) < 0$, then we further adjust ϵ so that $0 < \epsilon \leq -\frac{G(0)^{1-\alpha}}{2N'(0)}$. In any case, one has

$$Y(t) \geq \frac{1}{2} G(0)^{1-\alpha} > 0 \quad \text{for } t \in [0, T). \tag{2.25}$$

Finally, we show that

$$Y'(t) \geq C\epsilon^{1+\sigma} Y(t)^\eta \quad \text{for } t \in [0, T), \tag{2.26}$$

where

$$1 < \eta = \frac{1}{1-\alpha} < 2, \quad \sigma = 1 - \frac{2}{(1-2\alpha)(p+1)} > 0,$$

and $C > 0$ is a generic constant independent of ϵ . Notice that $\sigma > 0$ follows from the assumption $\alpha < \frac{p-1}{2(p+1)}$.

Now, if $N'(t) \leq 0$ for some $t \in [0, T)$, then for such value of t we have

$$Y(t)^\eta = [G(t)^{1-\alpha} + \epsilon N'(t)]^\eta \leq G(t) \tag{2.27}$$

and in this case, (2.23) and (2.27) yield

$$Y'(t) \geq 4\epsilon G(t) \geq 4\epsilon^{1+\sigma} G(t) \geq 4\epsilon^{1+\sigma} Y(t)^\eta.$$

Hence, (2.26) holds for all $t \in [0, T)$ for which $N'(t) \leq 0$. However, if $t \in [0, T)$ is such that $N'(t) > 0$, then showing the validity of (2.26) requires a little more effort. First, we note that $Y(t) = G(t)^{1-\alpha} + \epsilon N'(t) \leq G(t)^{1-\alpha} + N'(t)$, and so

$$Y(t)^\eta \leq 2^{\eta-1} [G(t) + N'(t)^\eta]. \tag{2.28}$$

We estimate $N'(t)^\eta$ as follows. By using Hölder and Young's inequality and noting that $1 < \eta < 2$, we obtain from (2.2) that

$$\begin{aligned} N'(t)^\eta &\leq 2^\eta \left(\|u_t(t)\|_2 \|u(t)\|_2 + \|v_t(t)\|_2 \|v(t)\|_2 \right)^\eta \\ &\leq C \left(\|u_t(t)\|_2^2 \|u(t)\|_{p+1}^\eta + \|v_t(t)\|_2^2 \|v(t)\|_{p+1}^\eta \right) \\ &\leq C \left(\|u_t(t)\|_2^2 + \|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \|v_t(t)\|_2^2 + \|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \right). \end{aligned} \tag{2.29}$$

Since $\eta = \frac{1}{1-\alpha}$ and $\sigma > 0$, it is easy to see that

$$\frac{2\eta}{(2-\eta)(p+1)} - 1 = \frac{2}{(1-2\alpha)(p+1)} - 1 = -\sigma < 0. \tag{2.30}$$

Therefore, by (2.3), (2.7), (2.30), and by recalling $\epsilon \leq G(0)$, we have

$$\begin{aligned} \|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} &= \left(\|u(t)\|_{p+1}^{p+1} \right)^{\frac{2\eta}{(2-\eta)(p+1)}} \leq CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}} \\ &\leq CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}-1} S(t) \leq CG(0)^{-\sigma} S(t) \leq C\epsilon^{-\sigma} S(t). \end{aligned} \tag{2.31}$$

Similarly,

$$\|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \leq C\epsilon^{-\sigma} S(t). \tag{2.32}$$

By (2.29) and (2.31)–(2.32) and noting $\epsilon^{-\sigma} > 1$, we obtain

$$\begin{aligned} N'(t)^\eta &\leq C \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \epsilon^{-\sigma} S(t) \right) \\ &\leq C\epsilon^{-\sigma} \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + S(t) \right). \end{aligned} \tag{2.33}$$

Finally, the estimates (2.23), (2.28) and (2.33) allow us to conclude that

$$\begin{aligned} Y'(t) &\geq C\epsilon \left[G(t) + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + S(t) \right] \geq C\epsilon [G(t) + \epsilon^\sigma N'(t)^\eta] \\ &\geq C\epsilon^{1+\sigma} [G(t) + N'(t)^\eta] \geq C\epsilon^{1+\sigma} Y(t)^\eta \end{aligned}$$

for all values of $t \in [0, T)$ for which $N'(t) > 0$. Hence, (2.26) is valid. By simple calculations, it follows from (2.25)–(2.26) that T is necessarily finite and

$$T < C\epsilon^{-(1+\sigma)} Y(0)^{-\frac{\sigma}{1-\alpha}} \leq C\epsilon^{-(1+\sigma)} G(0)^{-\alpha}. \tag{2.34}$$

As a result,

$$Y(t) = G(t)^{1-\alpha} + \epsilon N'(t) \rightarrow \infty \quad \text{as } t \rightarrow T^-. \tag{2.35}$$

It remains to show $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \rightarrow \infty$ as $t \rightarrow T^-$. Indeed, by the definition of $Y(t)$ and the first inequality in (2.33), one has

$$\begin{aligned} Y(t)^\eta &\leq 2^{\eta-1} [G(t) + \epsilon^\eta N'(t)^\eta] \\ &\leq 2^{\eta-1} \left[G(t) + \epsilon^\eta C \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \epsilon^{-\sigma} S(t) \right) \right]. \end{aligned} \tag{2.36}$$

By recalling (2.1), and by further adjusting ϵ so that $-\frac{1}{2} + \epsilon^\eta C \leq 0$, then (2.36) implies

$$Y(t)^\eta \leq 2^{\eta-1} [S(t) + C\epsilon^{\eta-\sigma} S(t)]. \tag{2.37}$$

However, by using the assumptions on the sources and employing Hölder's inequality, we have

$$\begin{aligned} S(t) &= \int_\Omega F(u(t), v(t)) dx + \int_\Gamma H(\gamma u(t)) d\Gamma \\ &\leq \frac{1}{c_1} \int_\Omega [u(t)f_1(u(t), v(t)) + v(t)f_2(u(t), v(t))] dx + \frac{1}{c_3} \int_\Gamma h(\gamma u(t)) \gamma u(t) d\Gamma \\ &\leq C \left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} + |\gamma u(t)|_{k+1}^{k+1} \right) \\ &\leq C \left(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1} \right), \end{aligned} \tag{2.38}$$

where we have used the fact $p < 5$ and $k < 3$, as mentioned in Remark 1.10. Consequently, by combining (2.37) and (2.38) one has

$$Y(t)^n \leq C \left(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1} \right),$$

and along with (2.35), we conclude $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \rightarrow \infty$ as $t \rightarrow T^-$. This completes the proof of Theorem 1.8. ■

3. Proof of Theorem 1.9

The proof of Theorem 1.9 goes along the same lines as the proof of Theorem 1.8; except for the estimate of the last term on the right-hand side of (2.16). Here, we shall utilize the following trace and interpolation theorems:

- Trace theorem (see e.g. [26]):

$$|\gamma u|_{q+1} \leq C \|u\|_{W^{s,q+1}(\Omega)}, \quad \text{where } s > \frac{1}{q+1}. \tag{3.1}$$

- Interpolation theorem [27]:

$$W^{1-\theta,r}(\Omega) = [H^1(\Omega), L^{p+1}(\Omega)]_{\theta}, \tag{3.2}$$

where $r = \frac{2(p+1)}{(1-\theta)(p+1)+2\theta}$, $\theta \in [0, 1]$, and as usual $[\cdot, \cdot]_{\theta}$ denotes the interpolation bracket.

We select θ such that

$$1 - \theta = \frac{1}{\beta(q+1)} > \frac{1}{q+1} \quad \text{for some } \frac{1}{q+1} < \beta < 1. \tag{3.3}$$

Additionally, we require that

$$r = \frac{2(p+1)}{(1-\theta)(p+1)+2\theta} \geq q+1. \tag{3.4}$$

Note $p > q$ since by assumption $p > 2q - 1 = q + (q - 1) \geq q$. So, inserting (3.3) into (3.4) yields the following restriction on β :

$$\beta \geq \frac{p-1}{2(p-q)} > 0. \tag{3.5}$$

However, since $q \geq 1$, and by assumption, $p > 2q - 1$, it follows that $1 > \frac{p-1}{2(p-q)} \geq \frac{1}{q+1}$. Thus, it is enough to impose the following restriction on β :

$$\frac{p-1}{2(p-q)} \leq \beta < 1. \tag{3.6}$$

Now, we turn our attention to the proof of Theorem 1.9.

Proof Under the above restrictions on the parameters, we first show that

$$|\gamma u|_{q+1} \leq C_1 \left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}} \right), \tag{3.7}$$

for some β satisfying (3.6), where C_1 is a generic constant.

In order to prove (3.7), we use (3.1)–(3.4) and Young’s inequality to obtain

$$\begin{aligned} |\gamma u|_{q+1} &\leq C \|u\|_{W^{1-\alpha}(\Omega)} \leq C \|u\|_{W^{1-\theta}(\Omega)} \leq C \|u\|_{1,\Omega}^{1-\theta} \|u\|_{p+1}^\theta \\ &= C \|u\|_{1,\Omega}^{\frac{1}{\beta(q+1)}} \|u\|_{p+1}^{1-\frac{1}{\beta(q+1)}} \leq C_1 \left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)}} \right). \end{aligned} \tag{3.8}$$

By comparing (3.7) and (3.8), it suffices to show that there exists β satisfying (3.6) such that $\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)} = \frac{(p+1)\beta}{q+1}$. We note that the latter is equivalent to $2(p+1)\beta^2 - 2(q+1)\beta - (p-1) = 0$. By the assumption $2q < p+1$, the positive root of the above quadratic equation satisfies:

$$\beta := \frac{2(q+1) + \sqrt{4(q+1)^2 + 8(p^2-1)}}{4(p+1)} < \frac{(p+3) + \sqrt{(3p+1)^2}}{4(p+1)} = 1. \tag{3.9}$$

Additionally, we must show that

$$\frac{2(q+1) + \sqrt{4(q+1)^2 + 8(p^2-1)}}{4(p+1)} \geq \frac{p-1}{2(p-q)}, \tag{3.10}$$

as required by (3.6). Indeed, by routine calculations, it is easy to see that (3.10) is equivalent to

$$(p-1)(p+1)^2(p-2q+1) \geq 0. \tag{3.11}$$

Obviously, (3.11) is valid since $p \geq 1$ and $p > 2q - 1$. Hence, (3.7), verified.

Now, we turn our attention to estimating the last term on the right-hand side of (2.16). First, we note that (2.15) yields

$$\|u(t)\|_{1,\Omega}^2 \leq 2S(t). \tag{3.12}$$

By Hölder’s inequality and the estimates (2.3), (3.7) and (3.12), we obtain

$$\begin{aligned} \left| \int_{\Gamma} g(\gamma u_t(t)) \gamma u(t) d\Gamma \right| &\leq b_3 \int_{\Gamma} |\gamma u(t)| |\gamma u_t(t)|^q d\Gamma \leq b_3 |\gamma u(t)|_{q+1} |\gamma u_t(t)|_{q+1}^q \\ &\leq b_3 C_1 \left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}} \right) |\gamma u_t(t)|_{q+1}^q \\ &\leq b_3 C_1 \left(2^{\frac{\beta}{q+1}} S(t)^{\frac{\beta}{q+1}} + c_0^{-\frac{\beta}{q+1}} S(t)^{\frac{\beta}{q+1}} \right) |\gamma u_t(t)|_{q+1}^q \\ &\leq K_4 S(t)^{\frac{\beta}{q+1}} |\gamma u_t(t)|_{q+1}^q, \end{aligned} \tag{3.13}$$

where $K_4 = b_3 C_1 \cdot \max\{2^{\frac{\beta}{q+1}}, c_0^{-\frac{\beta}{q+1}}\}$. In addition to the restriction on α in (2.4), we further require $\alpha < \frac{1-\beta}{q+1}$, so $\frac{\beta-1}{q+1} + \alpha < 0$. Thus, by using (2.6)–(2.7) and Young’s inequality, we can continue the estimate in (3.13) as follows.

$$\begin{aligned} \left| \int_{\Gamma} g(\gamma u_t(t)) \gamma u(t) d\Gamma \right| &\leq K_4 S(t)^{\frac{\beta-1}{q+1}} S(t)^{\frac{1}{q+1}} |\gamma u_t(t)|_{q+1}^q \\ &\leq G(t)^{\frac{\beta-1}{q+1}} \left(\delta_4 S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} |\gamma u_t(t)|_{q+1}^{q+1} \right) \end{aligned}$$

$$\begin{aligned} &\leq \delta_4 G(t)^{\frac{\beta-1}{q+1}} S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G'(t) G(t)^{-\alpha} G(t)^{\frac{\beta-1}{q+1} + \alpha} \\ &\leq \delta_4 G(0)^{\frac{\beta-1}{q+1}} S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G'(t) G(t)^{-\alpha} G(0)^{\frac{\beta-1}{q+1} + \alpha}, \end{aligned} \quad (3.14)$$

where $\delta_4 = \frac{\lambda}{6} G(0)^{\frac{1-\beta}{q+1}}$.

Now, instead of estimate (2.20) we use (3.14), and instead of (2.21) in Theorem 1.8, we choose $0 < \epsilon < 1$ small enough so that

$$\begin{aligned} L_1 = 1 - \alpha - 2\epsilon &\left(C_{\delta_1} K_1^{\frac{m+1}{m}} a_1^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{m+1} + \alpha} + C_{\delta_2} K_2^{\frac{r+1}{r}} a_2^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{r+1} + \alpha} \right. \\ &\left. + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G(0)^{\frac{\beta-1}{q+1} + \alpha} \right) \geq 0. \end{aligned}$$

After replacing L with L_1 in (2.23), the rest of the proof continues exactly as in the proof of Theorem 1.8. ■

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