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Global regularity for a rapidly rotating constrained convection model of tall columnar structure with weak dissipation

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Abstract

We study a three-dimensional fluid model describing rapidly rotating convection that takes place in tall columnar structures. The purpose of this model is to investigate the cyclonic and anticyclonic coherent structures. Global existence, uniqueness, continuous dependence on initial data, and large-time behavior of strong solutions are shown provided the model is regularized by a weak dissipation term.

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1. Introduction

1.1. The model

For the purpose of investigating the cyclonic and anticyclonic coherent structure in the Rayleigh-Bénard convection under the influence of a rapid rotation, Sprague et al. [10] (see also Julien et al. [7,6]) introduced and simulated the following asymptotically reduced system for rotationally constrained convection that takes place in a tall column:

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla_h w - \frac{\partial \phi}{\partial z} = \Gamma\theta + \frac{1}{Re}\Delta_h w, \tag{1.1}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla_h \omega - \frac{\partial w}{\partial z} = \frac{1}{Re}\Delta_h \omega, \tag{1.2}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla_h \theta + w\overline{w\theta} = \frac{1}{Pe}\Delta_h \theta, \tag{1.3}$$

$$\nabla_h \cdot \mathbf{u} = 0. \tag{1.4}$$

The above system is considered subject to periodic boundary conditions in $\mathbb{R}^3$ with fundamental periodic domain $\Omega = [0, L]^2 \times [0, 1]$. Here, $(u, v, w)^{tr}$ denotes the velocity vector field, and $\mathbf{u} = (u, v)^{tr}$ denotes the horizontal component of the velocity vector field. The unknown θ represents the fluctuation of the temperature such that the horizontal spatial mean $\overline{\theta}(z) = 0$, for every $z \in [0, 1]$. For a function f defined on Ω, the notation $\overline{f}$ stands for the horizontal mean $\overline{f}(z) = \frac{1}{L^2}\int_{[0,L]^2} f(x, y, z)dxdy$ for $z \in [0, 1]$. We denote the horizontal gradient by $\nabla_h = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^{tr}$ and denote the horizontal Laplacian by $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The unknown $\omega = \nabla_h \times \mathbf{u} = \partial_x v - \partial_y u$ represents the vertical component of the vorticity. As usual, the horizontal stream function ϕ is defined as $\phi = (-\Delta_h)^{-1}\omega$, with $\overline{\phi} = 0$. Also, a few dimensionless numbers appear in the model. Specifically, Re is the Reynolds number, Γ is the buoyancy number, and Pe is the Péclet number.

System (1.1)-(1.4) is an asymptotically reduced model derived from the three-dimensional Boussinesq equations governing buoyancy-driven rotational flow in tall columnar structures, by assuming that the ratio of the depth of the fluid layer to the horizontal scale is large, and that the rotation is fast. One may refer to [10] for the derivation of this model. In fact, when deriving model (1.1)-(1.4) from the 3D Boussinesq equations, the state variables, i.e., the velocity, pressure and temperature, are expanded in terms of the small parameter Ro, which stands for the Rossby number. For rapidly rotating flow, i.e., $Ro \ll 1$, the leading-order flow is horizontally divergence-free (see [10]). Roughly speaking, if the Rossby number is small in the Boussinesq equations, then the Coriolis force and the pressure gradient force are relatively large, which results in an equation that indicates that the leading order flow is in geostrophic balance: the pressure gradient force is balanced by the Coriolis effect. Then taking the curl of the geostrophic balance equation implies that the *horizontal* flow is divergence-free, namely, $\nabla_h \cdot \mathbf{u} = u_x + v_y = 0$.

We would like to stress that in model (1.1)-(1.4), the variable z stands for a *large* vertical scale, which is a scale different from x and y. More precisely, the original Boussinesq equations are considered in a tall column $[0, L]^2 \times [0, \ell]$, where the aspect ratio $\ell/L \gg 1$. We let $(x, y, \tilde{z}) \in [0, L]^2 \times [0, \ell]$ and rescale $\tilde{z} = \ell z$. Notice that the large vertical scale $z \in [0, 1]$, whereas the small vertical scale $\tilde{z} \in [0, \ell]$, for $\ell \gg 1$. Set $u(x, y, z) = \tilde{u}(x, y, \tilde{z})$, $v(x, y, z) = \tilde{v}(x, y, \tilde{z})$ and

$w(x, y, z) = \tilde{w}(x, y, \tilde{z})$, where $(\tilde{u}, \tilde{v}, \tilde{w})^{tr}$ represents the velocity vector field for the original Boussinesq equations. Note, the divergence-free condition of $(\tilde{u}, \tilde{v}, \tilde{w})^{tr}$ reads $\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0$, which implies that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\ell}\frac{\partial w}{\partial z} = 0$. Then, for $\ell \gg 1$, one can ignore the term $\frac{1}{\ell}\frac{\partial w}{\partial z}$ and obtain that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. In sum, the fast rotation and the tall columnar structure both lead to that the horizontal flow is divergence-free. Moreover, the absence of vertical diffusion in system (1.1)-(1.4) is also a consequence of the large aspect ratio of the fluid region. Furthermore, it is remarked in [10] that in the classical small-aspect-ratio (flat) regime, the strong stable stratification permits weak vertical motions only, while in the present large-aspect-ratio (tall) case, the unstable stratification permits substantial vertical motions.

The global regularity for system (1.1)-(1.4) is unknown. The main difficulty of analyzing (1.1)-(1.4) lies in the fact that the physical domain is three-dimensional, whereas the regularizing viscosity acts only on the horizontal variables, and the equations contain troublesome terms $\frac{\partial \phi}{\partial z}$ and $\frac{\partial w}{\partial z}$ involving the derivative in the vertical direction.

In this work, we regularize the convection model (1.1)-(1.4) by imposing a very weak vertical dissipation term $\epsilon^2 \frac{\partial^2 \phi}{\partial z^2}$ to the vorticity equation (1.2), namely, we consider the regularized system

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla_h w - \frac{\partial \phi}{\partial z} = \Gamma\theta + \frac{1}{Re}\Delta_h w, \tag{1.5}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla_h \omega - \frac{\partial w}{\partial z} = \frac{1}{Re}\Delta_h \omega + \epsilon^2 \frac{\partial^2 \phi}{\partial z^2}, \tag{1.6}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla_h \theta + w\overline{w\theta} = \frac{1}{Pe}\Delta_h \theta, \tag{1.7}$$

$$\nabla_h \cdot \mathbf{u} = 0. \tag{1.8}$$

The main goal of this paper is to prove the global regularity of system (1.5)-(1.8). We remark that, as a dissipation, $\epsilon^2 \frac{\partial^2 \phi}{\partial z^2}$ is much weaker than the vertical viscosity $\epsilon^2 \frac{\partial^2 \omega}{\partial z^2}$, since $\omega = -\Delta_h \phi$. The purpose of introducing and analyzing (1.5)-(1.8) is to shed some light on the global regularity problem for the 3D rotationally constrained convection model (1.1)-(1.4), a subject of future investigation. Notably, there is no physical meaning for the dissipation term $\epsilon^2 \frac{\partial^2 \phi}{\partial z^2}$, however, it can be viewed as a numerical dissipation. On the other hand, if one replaces the numerical dissipation $\epsilon^2 \frac{\partial^2 \phi}{\partial z^2}$ by the vertical viscous term $\epsilon^2 \frac{\partial^2 \omega}{\partial z^2}$, then the global regularity still holds but the analysis will be simpler. See Remark 3.2.

In order to prove the existence of strong solutions for (1.1)-(1.4), we introduce a "Galerkin-like" approximation scheme. In fact, the Galerkin-like system consists of a system of ODEs coupled with a PDE, and it is set up in the format of an iteration. This special Galerkin scheme represents a "novelty" of the paper.

For the original model (1.1)-(1.4), the *local* well-posedness of strong solutions can be established. But whether the local strong solution can be extended globally or forms singularity in finite time is unknown. Furthermore, system (1.1)-(1.4) does not contain any vertical diffusion, which results in a lack of necessary compactness, thus it is not clear how to establish global existence of weak solutions. Even with the help of the weak dissipation $\epsilon^2 \frac{\partial^2 \phi}{\partial z^2}$ in system (1.5)-(1.8), the global existence of weak solutions is still unknown, but this can be an interesting problem for future study.

It is worth mentioning that the three-dimensional Hasagawa-Mima equations [8,9], describing plasma turbulence, share a comparable structure with the convection model (1.1)-(1.4). Although the well-posedness problem for the 3D inviscid Hasagawa-Mima equations is still unsolved, in a recent work [3] we established the global well-posedness of strong solutions for a Hasegawa-Mima model with partial dissipation. Also, Cao et al. [2] showed the global well-posedness for an inviscid pseudo-Hasegawa-Mima model in three dimensions.

The paper is organized as follows. For the rest of section 1, we introduce suitable function spaces for solutions and provide some identities related to the nonlinearities of model (1.5)-(1.8). Then we state the main results, namely, the existence, uniqueness, continuous dependence on initial data, and large-time behavior of strong solutions to (1.5)-(1.8). Section 2 features some inequalities which are essential for our analysis. In section 3, we prove the existence of strong solutions by using a Galerkin-like approximation method. In section 4, we justify the uniqueness of strong solutions and the continuous dependence on initial data. Finally, we study the large-time behavior of solutions in section 5.

1.2. Preliminaries

Let $\Omega = [0, L]^2 \times [0, 1]$ be a three-dimensional fundamental periodic domain. The standard $L^p(\Omega)$ norm for periodic functions is denoted by $\|f\|_p = \left(\int_\Omega |f|^p dxdydz\right)^{\frac{1}{p}}$ for $p \geq 1$. As usual, the $L^2(\Omega)$ inner product of real-valued periodic functions f and g is defined by $(f, g) = \int_\Omega fgdxdydz$. Also $H^s(\Omega)$, $s \geq 0$, denotes the standard Sobolev spaces for periodic functions. In addition, we define the space

$$H_h^1(\Omega) = \left\{ f \in L^2(\Omega) : \int\limits_\Omega |\nabla_h f|^2 dxdydz < \infty \right\},$$

endowed with the norm $\|f\|_{H_h^1(\Omega)} = \left[\int_\Omega (|f|^2 + |\nabla_h f|^2) dxdydz\right]^{1/2}$.

Let $f \in H_h^1(\Omega)$ with zero horizontal mean, i.e. $\overline{f} = 0$, then the Poincaré inequality holds:

$$\|f\|_2^2 \leq \gamma \|\nabla_h f\|_2^2, \quad \text{where } \gamma = L^2/(4\pi^2). \tag{1.9}$$

For sufficiently smooth periodic functions $\mathbf{u}$, f and g, such that $\nabla_h \cdot \mathbf{u} = 0$, an integration by parts shows

$$(\mathbf{u} \cdot \nabla_h f, g) = -(\mathbf{u} \cdot \nabla_h g, f) \tag{1.10}$$

This implies

$$(\mathbf{u} \cdot \nabla_h f, f) = 0. \tag{1.11}$$

Note that the horizontal velocity $\mathbf{u}$, the vertical vorticity ω, and the horizontal stream function ϕ such that $\overline{\phi} = 0$ have the following relations:

$$\omega = \nabla_h \times \mathbf{u} = v_x - u_y, \quad \omega = -\Delta_h \phi, \quad \mathbf{u} = (\phi_y, -\phi_x)^{tr}. \tag{1.12}$$

It follows that

$$(\omega, \phi) = \|\mathbf{u}\|_2^2. \tag{1.13}$$

Also, $\|\omega\|_2 = \|\nabla_h \mathbf{u}\|_2$. In addition, by (1.10) and (1.12), we have

$$(\mathbf{u} \cdot \nabla_h f, \phi) = -(\mathbf{u} \cdot \nabla_h \phi, f) = 0, \tag{1.14}$$

for sufficiently regular functions $\mathbf{u}$, ϕ and f such that $\mathbf{u} = (\phi_y, -\phi_x)^{tr}$.

We remark that, since $\nabla_h \cdot \mathbf{u} = 0$ and $\omega = \nabla_h \times \mathbf{u} = v_x - u_y$, then $u = (-\Delta_h)^{-1}\omega_y$ and $v = \Delta_h^{-1}\omega_x$, if $\overline{\mathbf{u}} = 0$. Thus, the horizontal velocity $\mathbf{u}$ and the vertical component ω of the vorticity determine each other uniquely, provided $\nabla_h \cdot \mathbf{u} = 0$ and $\overline{\mathbf{u}} = 0$.

1.3. Main results

Before stating our main results, we shall give a precise definition of strong solutions for system (1.5)-(1.8). Let us first introduce a suitable function space for strong solutions to (1.5)-(1.8). Specifically, we define the following space of H^1 periodic functions on Ω with *horizontal* average zero:

$$V = \left\{(\mathbf{u}, w, \theta)^{tr} \in (H^1(\Omega))^4 : \nabla_h \cdot \mathbf{u} = 0,\ \ \overline{\mathbf{u}} = 0,\ \ \overline{w} = \overline{\theta} = 0\right\}. \tag{1.15}$$

Here, the fundamental periodic domain $\Omega = [0, L]^2 \times [0, 1]$. Also, recall the notation $\overline{f}$ stands for the horizontal mean of f, that is, $\overline{f} = \frac{1}{L^2}\int_{[0,L]^2} f(x, y, z)dxdy$ for $z \in [0, 1]$.

According to the derivation of model (1.1)-(1.4) in [10], all of the quantities including the velocity and temperature are “fluctuating” quantities about the horizontal mean, i.e., the original quantities subtracted by their horizontal means. Therefore, in the space V defined in (1.15), all quantities are demanded to have horizontal average zero. Also, we remark that these fluctuating quantities contain the essential information of the dynamics.

In addition, by assuming that $\overline{w_0} = 0$, $\overline{\omega_0} = 0$ and $\overline{\mathbf{u}_0} = 0$, and assuming that $\overline{\phi} = 0$ and $\overline{\theta} = 0$ for all $t \geq 0$, then we can derive formally from equations (1.5)-(1.8) that $\overline{w} = 0$, $\overline{\omega} = 0$ and $\overline{\mathbf{u}} = 0$ for all $t \geq 0$. Indeed, since $\nabla_h \cdot \mathbf{u} = 0$, it follows that $\overline{\mathbf{u} \cdot \nabla_h f} = 0$ for any smooth periodic function f defined on Ω. Therefore, by assuming $\overline{\phi} = 0$ and $\overline{\theta} = 0$ for all $t \geq 0$, we can take the horizontal average of each term in equation (1.5), to obtain that $\partial_t \overline{w} = 0$, which implies that $\overline{w} = 0$ for all $t \geq 0$ provided that $\overline{w_0} = 0$. Next, by taking the horizontal average of each term in equation (1.6) and using $\overline{w} = \overline{\phi} = 0$, we obtain that $\partial_t \overline{\omega} = 0$, which implies that $\overline{\omega} = 0$ for all $t \geq 0$ provided that $\overline{\omega_0} = 0$. Also, $\overline{\mathbf{u}} = 0$ since $\mathbf{u} = (\phi_y, -\phi_x)^{tr}$. Finally, by taking the horizontal mean of each term in equation (1.7) and using $\overline{w} = 0$, one has $\partial_t \overline{\theta} = 0$, which is consistent with the assumption that $\overline{\theta} = 0$ for all $t \geq 0$.

Definition 1.1. Let $T > 0$. Assume $(\mathbf{u}_0, w_0, \theta_0)^{tr} \in V$, thus $\omega_0 = \nabla_h \times \mathbf{u}_0 \in L^2(\Omega)$. We call $(\mathbf{u}, w, \theta)^{tr} \in V$ with $\omega \in L^2(\Omega)$ a strong solution for system (1.5)-(1.8) on $[0, T]$ if

(i) $\mathbf{u}$, w, θ and ω have the following regularity:

$$\begin{cases} \mathbf{u}, w, \theta \in L^\infty(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)); \\ \omega \in L^\infty(0,T; L^2(\Omega)) \cap C([0,T]; (H_h^1(\Omega))'); \\ \nabla_h \omega, \Delta_h w, \Delta_h \theta \in L^2(\Omega \times (0,T)); \\ \omega_z, \nabla_h w_z, \nabla_h \theta_z, \phi_{zz} \in L^2(\Omega \times (0,T)); \\ \mathbf{u}_t, w_t, \theta_t \in L^2(\Omega \times (0,T)); \\ \omega_t \in L^2(0,T; (H_h^1(\Omega))'). \end{cases} \tag{1.16}$$

(ii) equations (1.5)-(1.7) hold in the following function spaces respectively:

$$\begin{cases} w_t + \mathbf{u} \cdot \nabla_h w - \phi_z = \Gamma \theta + \frac{1}{Re} \Delta_h w, \ \text{ in } L^2(\Omega \times (0,T)); \\ \omega_t + \mathbf{u} \cdot \nabla_h \omega - w_z = \frac{1}{Re} \Delta_h \omega + \epsilon^2 \phi_{zz}, \ \text{ in } L^2(0,T; (H_h^1(\Omega))'); \\ \theta_t + \mathbf{u} \cdot \nabla_h \theta + w\overline{w\theta} = \frac{1}{Pe} \Delta_h \theta, \ \text{ in } L^2(\Omega \times (0,T)), \end{cases} \tag{1.17}$$

such that $\nabla_h \cdot \mathbf{u} = 0$, $\omega = \nabla_h \times \mathbf{u} = -\Delta_h \phi$, with $\overline{\phi} = 0$.

(iii) $\mathbf{u}(0) = \mathbf{u}_0$, $w(0) = w_0$, $\omega(0) = \omega_0$, $\theta(0) = \theta_0$.

Now we are ready to state the main results of the manuscript. Our first theorem is concerned with the global existence and uniqueness of strong solutions as well as the continuous dependence on initial data.

Theorem 1.2. *Let $T > 0$. Assume initial data $(\mathbf{u}_0, w_0, \theta_0)^{tr} \in V$, thus $\omega_0 = \nabla_h \times \mathbf{u}_0 \in L^2(\Omega)$. Then system (1.5)-(1.8) admits a unique strong solution $(\mathbf{u}, w, \theta) \in V$ with $\omega \in L^2(\Omega)$ on $[0,T]$ in the sense of Definition 1.1. Moreover, the energy identity is valid for every $t \in [0,T]$:*

$$\begin{aligned} &\frac{1}{2}\left(\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2 + \|\theta(t)\|_2^2\right) + \int_0^t \|\overline{w\theta}\|_2^2 ds \\ &+ \int_0^t \left[\frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h \mathbf{u}\|_2^2\right) + \frac{1}{Pe}\|\nabla_h \theta\|_2^2 + \epsilon^2 \|\phi_z\|_2^2\right] ds \\ &= \frac{1}{2}\left(\|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 + \|\theta_0\|_2^2\right) + \Gamma \int_0^t (\theta, w) ds. \end{aligned} \tag{1.18}$$

Also, if $(\mathbf{u}_0^n, w_0^n, \theta_0^n)^{tr}$ is a bounded sequence of initial data in V and $\omega_0^n = \nabla_h \times \mathbf{u}_0^n$ is a bounded sequence in $L^2(\Omega)$ such that $(\mathbf{u}_0^n, w_0^n, \theta_0^n)^{tr}$ converges to $(\mathbf{u}_0, w_0, \theta_0)^{tr}$ with respect to the $L^2(\Omega)$-norm and ω_0^n converges to ω_0 in $(H_h^1(\Omega))'$, then the corresponding strong solution $(\mathbf{u}^n, w^n, \theta^n)^{tr}$ converges to $(\mathbf{u}, w, \theta)^{tr}$ in $C([0,T]; (L^2(\Omega))^4)$, and ω^n converges to ω in $C([0,T]; (H_h^1(\Omega))')$.

The next result is concerned with the large-time behavior of strong solutions for system (1.5)-(1.8). It shows the energy decays to zero exponentially in time. Also, the L^2-norm of ω

as well as the L^2-norm of the horizontal gradient of w and θ approach zero exponentially fast. However, the L^2-norm of the vertical derivative of $(\mathbf{u}, w, \theta)^{tr}$ grows at most exponentially in time. Recall we have defined $\gamma = L^2/(4\pi^2)$ in (1.9).

Theorem 1.3. *Let $\kappa \geq 1$ such that $Pe \neq 2\kappa Re$. Assume $(\mathbf{u}, w, \theta)^{tr} \in V$ with $\omega \in L^2(\Omega)$ is a global strong solution for system (1.5)-(1.8) in the sense of Definition 1.1. Then, for all $t \geq 0$,*

$$\|\theta(t)\|_2^2 \leq e^{-\frac{2}{\gamma Pe}t}\|\theta_0\|_2^2, \tag{1.19}$$

$$\|\mathbf{u}(t)\|_2^2 + \|w(t)\|_2^2 \leq e^{-\frac{1}{\kappa\gamma Re}t}\left(\|\mathbf{u}_0\|_2^2 + \|w_0\|_2^2\right) + C\left(e^{-\frac{2}{\gamma Pe}t} + e^{-\frac{1}{\kappa\gamma Re}t}\right)\|\theta_0\|_2^2. \tag{1.20}$$

In addition, for large t,

$$\begin{aligned}&\|\omega(t)\|_2^2 + \|\nabla_h w(t)\|_2^2 + \|\nabla_h\theta(t)\|_2^2 \\ &\leq \left(e^{-\frac{2}{\gamma Pe}t} + e^{-\frac{1}{\kappa\gamma Re}t}\right)C(\|\mathbf{u}_0\|_2, \|w_0\|_2, \|\theta_0\|_2, \|\partial_z\theta_0\|_2).\end{aligned} \tag{1.21}$$

Moreover, for all $t \geq 0$,

$$\|\mathbf{u}_z(t)\|_2^2 + \|w_z(t)\|_2^2 + \|\theta_z(t)\|_2^2 \leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1})e^{C\left(\|\theta_0\|_2^2 + \|\partial_z\theta_0\|_2^2 + 1\right)t}. \tag{1.22}$$

2. Auxiliary inequalities

In this section, we provide some inequalities which are essential for analyzing model (1.5)-(1.8). The first one is an anisotropic Ladyzhenskaya inequality which will be used repeatedly in this manuscript.

Lemma 2.1. *Let $f \in H^1(\Omega)$, $g \in H_h^1(\Omega)$ and $h \in L^2(\Omega)$. Then*

$$\int_\Omega |fgh|dxdydz \leq C\left(\|f\|_2 + \|\nabla_h f\|_2\right)^{\frac{1}{2}}\left(\|f\|_2 + \|f_z\|_2\right)^{\frac{1}{2}}\|g\|_2^{\frac{1}{2}}\left(\|g\|_2 + \|\nabla_h g\|_2\right)^{\frac{1}{2}}\|h\|_2.$$

We have proved Lemma 2.1 in [3]. Also, a similar inequality can be found in [4].

The next inequality is derived from the Agmon's inequality.

Lemma 2.2. *Assume $f,\ f_z \in L^2(\Omega)$, then*

$$\sup_{z\in[0,1]}\int_{[0,L]^2}|f(x,y,z)|^2dxdy \leq C\|f\|_2\left(\|f\|_2 + \|f_z\|_2\right). \tag{2.1}$$

Proof. Recall the Agmon's inequality in one dimension (cf. [1,5,11]):

$$\|\psi\|_{L^\infty([0,1])} \leq C\|\psi\|_{L^2([0,1])}^{1/2}\|\psi\|_{H^1([0,1])}^{1/2}, \quad \text{for any } \psi \in H^1([0,1]).$$

Then, we have, for a.e. $z \in [0,1]$,

$$
\begin{aligned}
&\int\limits_{[0,L]^2} |f(x,y,z)|^2 dxdy \\
&\le C \int\limits_{[0,L]^2} \left(\int\limits_0^1 |f|^2 dz \right)^{1/2} \left(\int\limits_0^1 (|f|^2 + |f_z|^2) dz \right)^{1/2} dxdy \\
&\le C \|f\|_2 \left(\|f\|_2 + \|f_z\|_2 \right),
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality in the last step. □

Next, we derive an elementary Gronwall-type inequality which will be used to deal with the temperature equation (1.7) in section 3.2.7.

Lemma 2.3. *Given continuous non-negative functions* $\eta,\ f,\ g,\ h : [0,\infty) \to [0,\infty)$ *such that* $\eta \in C^1([0,\infty))$. *Suppose*

$$
\eta \frac{d\eta}{dt} + h \le f + g\eta, \ \textit{for all } t \ge 0, \tag{2.2}
$$

then

$$
\eta^2(t) + 2\int\limits_0^t h(s)ds \le C\left[\eta^2(0) + \int\limits_0^t f(s)ds + \left(\int\limits_0^t g(s)ds \right)^2 \right], \ \textit{for all } t \ge 0. \tag{2.3}
$$

Proof. Set $\xi(t) = \eta(t) - \int_0^t g(s)ds$. Since $\eta \in C^1$ and g is continuous, then ξ is also C^1.

Let us fix an arbitrary time $T > 0$.

If $\xi(T) \le 0$, then

$$
\eta(T) \le \int\limits_0^T g(s)ds. \tag{2.4}
$$

If $\xi(T) > 0$, then we consider the following two cases.

Case 1: $\xi(t) > 0$ *for all* $t \in [0,T]$.

By (2.2) and the assumption that h is a non-negative function, one has $\eta\eta' \le f + g\eta$, that is,

$$
\xi\xi' + \xi' \int\limits_0^t g(s)ds \le f, \ \text{ for all } t \in [0,T]. \tag{2.5}
$$

On the one hand, if $\xi' \ge 0$ on some interval $[a,b] \subset [0,T]$, and since g is non-negative, then (2.5) implies that $\xi\xi' \le f$ on $[a,b]$. Thus

$$\xi^2(t) \le \xi^2(a) + 2\int_a^t f(s)ds, \quad \text{for all } t \in [a,b] \subset [0,T]. \tag{2.6}$$

On the other hand, if $\xi' \le 0$ on some interval $[a,b] \subset [0,T]$, then $\xi(t) \le \xi(a)$ for all $t \in [a,b]$, thus $\xi^2(t) \le \xi^2(a)$ for all $t \in [a,b]$, as $\xi(t)$ is positive on $[0,T]$. Therefore, (2.6) also holds. As a result, we have $\xi^2(t) \le \xi^2(0) + 2\int_0^t f(s)ds$ for all $t \in [0,T]$, which implies

$$\xi(t) \le \xi(0) + \left(2\int_0^t f(s)ds\right)^{1/2}, \quad \text{for all } t \in [0,T]. \tag{2.7}$$

Since $\eta(t) = \xi(t) + \int_0^t g(s)ds$ and along with (2.7), we obtain

$$\begin{aligned}\eta(t) &\le \xi(0) + \left(2\int_0^t f(s)ds\right)^{1/2} + \int_0^t g(s)ds \\ &= \eta(0) + \left(2\int_0^t f(s)ds\right)^{1/2} + \int_0^t g(s)ds, \quad \text{for all } t \in [0,T].\end{aligned} \tag{2.8}$$

Case 2: $\xi(t)$ is not always positive on $[0,T]$.

In this case, since $\xi(T) > 0$ and ξ is C^1, there exists $t^* \in [0,T)$ such that $\xi(t^*) = 0$ and $\xi(t) > 0$ for all $t \in (t^*, T]$. Then, by using similar estimates as (2.5)-(2.7) from Case 1, we obtain

$$\begin{aligned}\xi(t) &\le \xi(t^*) + \left(2\int_{t^*}^t f(s)ds\right)^{1/2} \\ &= \left(2\int_{t^*}^t f(s)ds\right)^{1/2} \le \left(2\int_0^t f(s)ds\right)^{1/2}, \quad \text{for all } t \in [t^*, T].\end{aligned} \tag{2.9}$$

Since $\eta(t) = \xi(t) + \int_0^t g(s)ds$ and together with (2.9), one has

$$\eta(t) \le \left(2\int_0^t f(s)ds\right)^{1/2} + \int_0^t g(s)ds, \quad \text{for all } t \in [t^*, T]. \tag{2.10}$$

Notice that (2.8) and (2.10) hold at $t = T$. As a result, for both Case 1 and Case 2, the following inequality is valid:

$$\eta(T) \le \eta(0) + \left(2\int_0^T f(s)ds\right)^{1/2} + \int_0^T g(s)ds. \tag{2.11}$$

Due to (2.4) and (2.11), we see that, no matter what the sign of $\xi(T)$ is, (2.11) always holds. Furthermore, since $T > 0$ is an arbitrary time, (2.11) can be expressed as

$$\eta(t) \le \eta(0) + \left(2\int_0^t f(s)ds\right)^{1/2} + \int_0^t g(s)ds, \quad \text{for all } t \ge 0. \tag{2.12}$$

Next, we substitute (2.12) into the right-hand side of (2.2). Then

$$\frac{1}{2}\frac{d}{dt}[\eta^2(t)] + h(t) \le f(t) + g(t)\left[\eta(0) + \left(2\int_0^t f(s)ds\right)^{1/2} + \int_0^t g(s)ds\right], \tag{2.13}$$

for all $t \ge 0$. Integrating (2.13) over $[0, t]$ yields

$$\begin{aligned}
&\frac{1}{2}\eta^2(t) + \int_0^t h(s)ds \\
&\le \frac{1}{2}\eta^2(0) + \int_0^t f(s)ds + \int_0^t \left[g(s)\left(\eta(0) + \left(2\int_0^s f(\tau)d\tau\right)^{1/2} + \int_0^s g(\tau)d\tau\right)\right] ds \\
&\le \frac{1}{2}\eta^2(0) + \int_0^t f(s)ds + \left(\eta(0) + \left(2\int_0^t f(\tau)d\tau\right)^{1/2} + \int_0^t g(\tau)d\tau\right)\int_0^t g(s)ds \\
&\le C\left[\eta^2(0) + \int_0^t f(s)ds + \left(\int_0^t g(s)ds\right)^2\right],
\end{aligned}$$

where in the last step we use Young's inequality. □

Finally, we state a well-known uniform Gronwall Lemma. The proof can be found, e.g., in [11].

Lemma 2.4. *Let g, h, η be three positive locally integrable functions on $[0, \infty)$ such that η' is locally integrable, and which satisfy*

$$\frac{d\eta}{dt} \le g\eta + h, \quad \textit{for } t \ge 0.$$

Then,

$$\eta(t+1) \le e^{\int_t^{t+1} g(s)ds} \left(\int_t^{t+1} \eta(s)ds + \int_t^{t+1} h(s)ds \right), \quad for\ all\ t \ge 0.$$

3. Existence of strong solutions

Our strategy for proving the existence of strong solutions for system (1.5)-(1.8) is a "modified" Galerkin method. The *a priori* estimate for θ involves L^∞ norm in the vertical variable, which is not easy to obtain with the standard Galerkin approximation scheme. To overcome the difficulty, we propose a "Galerkin-like" system, which consists of a Galerkin approximation for velocity equations only, coupled with a PDE for the temperature.

3.1. Galerkin-like approximation system

We assume initial data $\mathbf{u}_0, w_0, \theta_0 \in H^1(\Omega)$ with $\overline{\mathbf{u}_0} = 0$, $\overline{w_0} = \overline{\theta_0} = 0$, and $\omega_0 = \nabla_h \times \mathbf{u}_0 \in L^2(\Omega)$.

Let $e_{\mathbf{j}} = \exp(2\pi i[(j_1 x + j_2 y)/L + j_3 z])$ for $\mathbf{j} = (j_1, j_2, j_3)^{tr} \in \mathbb{Z}^3$, which form a basis of the $L^2(\Omega)$ space of periodic functions in $\Omega = [0, L]^2 \times [0, 1]$. For $m \in \mathbb{N}$, denote by $P_m\left(L^2(\Omega)\right)$ the subspace of $L^2(\Omega)$ spanned by $\{e_{\mathbf{j}}\}_{|\mathbf{j}|\le m}$. Also, for an $L^2(\Omega)$ function $f = \sum_{\mathbf{j}\in\mathbb{Z}^3} f_{\mathbf{j}} e_{\mathbf{j}}$, where $f_{\mathbf{j}} = (f, e_{\mathbf{j}})$, we denote by $P_m f = \sum_{|\mathbf{j}|\le m} f_{\mathbf{j}} e_{\mathbf{j}}$ the orthogonal projection.

In order to prove the existence of strong solutions for system (1.5)-(1.8), we introduce the following "Galerkin-like" approximation system, for $m \ge 2$,

$$\partial_t w_m + P_m(\mathbf{u}_m \cdot \nabla_h w_m) - \partial_z \phi_m = \Gamma P_m \theta^{(m-1)} + \frac{1}{Re}\Delta_h w_m, \tag{3.1}$$

$$\partial_t \omega_m + P_m(\mathbf{u}_m \cdot \nabla_h \omega_m) - \partial_z w_m = \frac{1}{Re}\Delta_h \omega_m + \epsilon^2 \partial_{zz}\phi_m, \tag{3.2}$$

$$\nabla_h \cdot \mathbf{u}_m = 0, \tag{3.3}$$

$$\omega_m = \nabla_h \times \mathbf{u}_m = -\Delta_h \phi_m, \text{ with } \overline{\phi_m} = 0, \tag{3.4}$$

$$\partial_t \theta^{(m)} + \mathbf{u}_m \cdot \nabla_h \theta^{(m)} + w_m \overline{w_m \theta^{(m)}} = \frac{1}{Pe}\Delta_h \theta^{(m)}, \tag{3.5}$$

where $\omega_m, \mathbf{u}_m, \phi_m, w_m \in P_m\left(L^2(\Omega)\right)$, with the initial condition

$$\omega_m(0) = P_m\omega_0, \ \ \mathbf{u}_m(0) = P_m\mathbf{u}_0, \ \ w_m(0) = P_m w_0, \ \ \theta^{(m)}(0) = \theta_0. \tag{3.6}$$

Since (3.1) includes the term $\theta^{(m-1)}$, for $m \ge 2$, we have to specify $\theta^{(1)}$. Here, we let $\theta^{(1)}$ satisfy the heat equation

$$\partial_t \theta^{(1)} - \frac{1}{Pe}\Delta_h \theta^{(1)} = 0, \ \text{ with } \theta^{(1)}(0) = \theta_0. \tag{3.7}$$

For a given $\theta^{(m-1)}$, velocity equations (3.1)-(3.4) are genuine Galerkin approximation at level m, which can be regarded as a system of ODEs, whereas the temperature equation (3.5) is a PDE. On the one hand, ω_m, $\mathbf{u}_m$, w_m are in the subspace $P_m(L^2(\Omega))$, namely, they are finite linear combination of Fourier modes. On the other hand, we do not demand $\theta^{(m)}$ to be finite combination of Fourier modes, so the superscript m is adopted to emphasize the distinction between $\theta^{(m)}$ and $(\omega_m, \mathbf{u}_m, w_m)^{tr}$.

It is important to notice that the "Galerkin-like" system (3.1)-(3.6) is set up in the format of an iteration. Let $T > 0$. We claim, for any $m \geq 2$, system (3.1)-(3.6) has a unique solution on $[0, T]$. This can be seen by an induction process described as follows. To begin the induction, a function $\theta^{(1)} \in C([0,T]; H^1(\Omega))$ is given satisfying (3.7). Now, we assume $\theta^{(m-1)} \in C([0,T]; H^1(\Omega))$ is known for an $m \geq 2$, and show that system (3.1)-(3.6) possesses a unique solution $(\omega_m, \mathbf{u}_m, w_m, \theta^{(m)})^{tr}$ on $[0, T]$. Indeed, since the velocity equations (3.1)-(3.4) form a system of ODEs with quadratic nonlinearities, by the standard theory of ordinary differential equations, a unique classical solution $(\omega_m, \mathbf{u}_m, w_m)^{tr}$ for (3.1)-(3.4) exists for a short time. Furthermore, one can show that the L^2 norm of $(\mathbf{u}_m, w_m)^{tr}$ has a bound independent of time (see (3.15) below), thus $(\omega_m, \mathbf{u}_m, w_m)^{tr}$ can be extended to $[0, T]$. Because $\mathbf{u}_m$ and w_m have finitely many Fourier modes, they are analytic in space. Next we input $\mathbf{u}_m$ and w_m into the temperature equation (3.5) and solve for $\theta^{(m)}$. At this stage, $\mathbf{u}_m$ and w_m are known smooth functions, thus (3.5) is a *linear* PDE, which has a unique solution

$$\theta^{(m)} \in C([0,T]; H^1(\Omega)) \text{ with } \partial_t \theta^{(m)}, \Delta_h \theta^{(m)}, \nabla_h \partial_z \theta^{(m)} \in L^2(\Omega \times (0,T)), \tag{3.8}$$

$$\text{and } \partial_t \partial_z \theta^{(m)} \in L^2(0,T; (H_h^1(\Omega))'), \tag{3.9}$$

so that (3.5) holds in the space $L^2(\Omega \times (0, T))$. Then, we can put $\theta^{(m)}$ back into (3.1) to repeat the procedure to obtain $(\omega_{m+1}, \mathbf{u}_{m+1}, w_{m+1}, \theta^{(m+1)})^{tr}$. In conclusion, given $\theta^{(1)}$ satisfying (3.7), by induction, the "Galerkin-like" system (3.1)-(3.6) has a unique solution $(\omega_m, \mathbf{u}_m, w_m, \theta^{(m)})^{tr}$ on $[0, T]$, for any $m \geq 2$.

We aim to show that the $H^1(\Omega)$ norm of $(\mathbf{u}_m, w_m, \theta^{(m)})^{tr}$ is bounded on $[0, T]$ uniformly in m, and there exists a subsequence converging to a solution $(\mathbf{u}, w, \theta)^{tr}$ of system (1.5)-(1.8).

Remark 3.1. By assuming $\overline{\mathbf{u}_0} = 0$, $\overline{w_0} = 0$ and $\overline{\theta_0} = 0$, we have

$$\overline{\mathbf{u}_m} = 0, \ \ \overline{w_m} = 0, \ \ \overline{\theta^{(m)}} = 0, \ \ \text{for all } t \in [0,T], \ m \geq 2. \tag{3.10}$$

Indeed, since (3.3)-(3.4) hold, it is required that $\overline{\omega_m} = 0$, $\overline{\mathbf{u}_m} = 0$, and $\overline{\phi_m} = 0$ for all $m \geq 2$. To see that $\overline{w_m} = 0$ and $\overline{\theta^{(m)}} = 0$ for all $m \geq 2$, we use induction. First, note that $\overline{\theta^{(1)}} = 0$, due to (3.7) and $\overline{\theta_0} = 0$. Now, we assume $\overline{\theta^{(m-1)}} = 0$ for an $m \geq 2$, and show $\overline{w_m} = 0$, $\overline{\theta^{(m)}} = 0$. In fact, by taking the horizontal mean of each term of (3.1) and using $\nabla_h \cdot \mathbf{u}_m = 0$, we obtain $\partial_t \overline{w_m} = 0$. Then, because $\overline{w_m}(0) = \overline{P_m w_0} = 0$, it follows that $\overline{w_m} = 0$. Next we take the horizontal mean of each term of the temperature equation (3.5) to get $\partial_t \overline{\theta^{(m)}} = 0$, which implies $\overline{\theta^{(m)}} = 0$, since $\overline{\theta^{(m)}}(0) = \overline{\theta_0} = 0$. Finally, to check whether (3.10) is consistent with equation (3.2), we take the horizontal mean on (3.2), then all terms vanish, if (3.10) holds.

3.2. Uniform bound for $(\mathbf{u}_m, w_m, \theta^{(m)})^{tr}$ *in* $H^1(\Omega)$

This section is devoted to proving that $(\mathbf{u}_m, w_m, \theta^{(m)})^{tr}$ has a uniform bound in $L^\infty(0,T; H^1(\Omega))$ independent of m. It implies that ω_m is uniformly bounded in $L^\infty(0,T; L^2(\Omega))$. The calculations in this section are legitimate, because solutions for (3.1)-(3.6) are sufficiently regular. More precisely, ω_m, $\mathbf{u}_m$ and w_m are trigonometric polynomials satisfying (3.1)-(3.4) in the classic sense, while $\theta^{(m)}$ has regularity (3.8)-(3.9) so that equation (3.5) holds in $L^2(\Omega \times (0, T))$.

3.2.1. Estimate for $\|\theta^{(m)}\|_2^2$

For any $m \geq 2$, we multiply (3.3) with $\theta^{(m)}$ and then integrate it over $\Omega \times [0, t]$ to get

$$\frac{1}{2}\|\theta^{(m)}(t)\|_2^2 + \int_0^t \left(\frac{1}{Pe}\|\nabla_h \theta^{(m)}\|_2^2 + \|\overline{w_m \theta^{(m)}}\|_2^2 \right) ds = \frac{1}{2}\|\theta^{(m)}(0)\|_2^2 = \frac{1}{2}\|\theta_0\|_2^2, \quad (3.11)$$

for all $t \in [0, T]$. Also, for $\theta^{(1)}$, we obtain from (3.7) that

$$\frac{1}{2}\|\theta^{(1)}(t)\|_2^2 + \int_0^t \frac{1}{Pe}\|\nabla_h \theta^{(1)}\|_2^2 ds = \frac{1}{2}\|\theta^{(1)}(0)\|_2^2 = \frac{1}{2}\|\theta_0\|_2^2, \quad \text{for all } t \in [0, T]. \quad (3.12)$$

3.2.2. Estimate for $\|w_m\|_2^2 + \|\mathbf{u}_m\|_2^2$

Taking the $L^2(\Omega)$ inner product of equations (3.1)-(3.2) with $(w_m, \phi_m)^{tr}$ shows

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|w_m\|_2^2 + \|\mathbf{u}_m\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w_m\|_2^2 + \|\nabla_h \mathbf{u}_m\|_2^2\right) + \epsilon^2\|\partial_z \phi_m\|_2^2 \\ &= \Gamma(\theta^{(m-1)}, w_m) \end{aligned} \quad (3.13)$$

where we have used identities (1.11), (1.13) and (1.14).

Since the horizontal mean $\overline{w_m} = 0$ by (3.10), one has the Poincaré inequality $\|w_m\|_2 \leq \gamma\|\nabla_h w_m\|_2$. Then $\Gamma(\theta^{(m-1)}, w_m) \leq \Gamma\|\theta^{(m-1)}\|_2\|w_m\|_2 \leq \frac{1}{2Re}\|\nabla_h w_m\|_2^2 + C\|\theta^{(m-1)}\|_2^2$. As a result,

$$\frac{d}{dt}\left(\|w_m\|_2^2 + \|\mathbf{u}_m\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w_m\|_2^2 + \|\nabla_h \mathbf{u}_m\|_2^2\right) + \epsilon^2\|\partial_z \phi_m\|_2^2 \leq C\|\theta^{(m-1)}\|_2^2. \quad (3.14)$$

Integrating (3.14) over $[0, t]$, we obtain, for $m \geq 2$,

$$\begin{aligned} &\|w_m(t)\|_2^2 + \|\mathbf{u}_m(t)\|_2^2 + \int_0^t \left(\frac{1}{Re}\left(\|\nabla_h w_m\|_2^2 + \|\nabla_h \mathbf{u}_m\|_2^2\right) + \epsilon^2\|\partial_z \phi_m\|_2^2 \right) ds \\ &\leq \|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 + C\int_0^t \|\theta^{(m-1)}\|_2^2 ds \leq \|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 + C\|\theta_0\|_2^2, \end{aligned} \quad (3.15)$$

for all $t \in [0, T]$, where the last inequality is due to (3.11) and (3.12).

3.2.3. Estimate for $\|\omega_m\|_2^2$

Taking the inner product of (3.2) with ω_m yields

$$\frac{1}{2}\frac{d}{dt}\|\omega_m\|_2^2 + \frac{1}{Re}\|\nabla_h \omega_m\|_2^2 + \epsilon^2\|\partial_z \mathbf{u}_m\|_2^2 = (\partial_z w_m, \omega_m), \quad (3.16)$$

where (1.11) and (1.13) have been used. Thanks to (1.12), after integration by parts, we have

$$
\begin{aligned}
&(\partial_z w_m, \omega_m) = \int_\Omega \partial_z w_m(-\Delta_h \phi_m) dxdydz = -\int_\Omega \nabla_h w_m \cdot \nabla_h \partial_z \phi_m dxdydz \\
&\le \|\nabla_h w_m\|_2 \|\nabla_h \partial_z \phi_m\|_2 = \|\nabla_h w_m\|_2 \|\partial_z \mathbf{u}_m\|_2 \le \frac{\epsilon^2}{2}\|\partial_z \mathbf{u}_m\|_2^2 + \frac{1}{2\epsilon^2}\|\nabla_h w_m\|_2^2. \quad (3.17)
\end{aligned}
$$

Combining (3.16) and (3.17) implies

$$
\frac{d}{dt}\|\omega_m\|_2^2 + \frac{2}{Re}\|\nabla_h \omega_m\|_2^2 + \epsilon^2 \|\partial_z \mathbf{u}_m\|_2^2 \le \frac{1}{\epsilon^2}\|\nabla_h w_m\|_2^2. \tag{3.18}
$$

By integrating (3.18) over the interval $[0, t]$, we obtain, for $m \ge 2$,

$$
\begin{aligned}
&\|\omega_m(t)\|_2^2 + \int_0^t \left(\frac{2}{Re}\|\nabla_h \omega_m\|_2^2 + \epsilon^2 \|\partial_z \mathbf{u}_m\|_2^2\right) ds \le \|\omega_m(0)\|_2^2 + \frac{1}{\epsilon^2}\int_0^t \|\nabla_h w_m\|_2^2 ds \\
&\le \|\omega_0\|_2^2 + C\left(\|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 + \|\theta_0\|_2^2\right), \quad \text{for all } t \in [0, T], \quad (3.19)
\end{aligned}
$$

where the last inequality is due to (3.15).

3.2.4. Estimate for $\|\nabla_h w_m\|_2^2$

Taking the inner product of (3.1) with $-\Delta_h w_m$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla_h w_m\|_2^2 + \frac{1}{Re}\|\Delta_h w_m\|_2^2 \\
&\le \int_\Omega |(\mathbf{u}_m \cdot \nabla_h w_m)\Delta_h w_m| dxdydz + \|\partial_z \phi_m\|_2 \|\Delta_h w_m\|_2 + \Gamma \|\theta^{(m-1)}\|_2 \|\Delta_h w_m\|_2 \\
&\le C\|\omega_m\|_2^{1/2}(\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2}\|\nabla_h w_m\|_2^{1/2}\|\Delta_h w_m\|_2^{3/2} \\
&\quad + \|\partial_z \phi_m\|_2 \|\Delta_h w_m\|_2 + \Gamma \|\theta^{(m-1)}\|_2 \|\Delta_h w_m\|_2, \quad (3.20)
\end{aligned}
$$

where we have used Lemma 2.1 to establish the last inequality. Then, employing the Young's inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla_h w_m\|_2^2 + \frac{1}{Re}\|\Delta_h w_m\|_2^2 \\
&\le C\|\omega_m\|_2^2(\|\mathbf{u}_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2)\|\nabla_h w_m\|_2^2 + C\left(\|\partial_z \phi_m\|_2^2 + \|\theta^{(m-1)}\|_2^2\right). \quad (3.21)
\end{aligned}
$$

Thanks to the Gronwall's inequality, we have, for $m \ge 2$,

$$
\|\nabla_h w_m(t)\|_2^2 + \frac{1}{Re}\int_0^t \|\Delta_h w_m\|_2^2 ds
$$

$$
\begin{aligned}
&\leq \left(\|\nabla_h w_m(0)\|_2^2 + C \int_0^t \left(\|\partial_z \phi_m\|_2^2 + \|\theta^{(m-1)}\|_2^2 \right) ds \right) e^{\int_0^t C \|\omega_m\|_2^2 (\|\mathbf{u}_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2) ds} \\
&\leq C(\|\nabla_h w_0\|_2, \|\omega_0\|_2, \|\theta_0\|_2), \quad \text{for all } t \in [0,T], \qquad (3.22)
\end{aligned}
$$

where the last inequality is due to estimates (3.11), (3.12), (3.15) and (3.19).

3.2.5. Estimate for $\|\overline{|\theta^{(m)}|^2}\|_{L^\infty}$

We multiply (3.3) by $\theta^{(m)}$ and integrate the result with respect to horizontal variables over $[0,L]^2$. Recall the notation for the horizontal mean $\overline{f} = \frac{1}{L^2}\int_{[0,L]^2} f\,dxdy$. Since $\nabla_h \cdot \mathbf{u}_m = 0$, it follows that

$$
\frac{1}{2}\frac{\partial}{\partial t}\overline{|\theta^{(m)}|^2}(z) + \frac{1}{Pe}\overline{|\nabla_h \theta^{(m)}|^2}(z) + \left(\overline{w_m \theta^{(m)}}\right)^2(z) = 0, \qquad (3.23)
$$

for a.e. $z \in [0,1]$. Integrating (3.23) over $[0,t]$ gives us

$$
\begin{aligned}
&\frac{1}{2}\overline{|\theta^{(m)}|^2}(z,t) + \int_0^t \left[\frac{1}{Pe}\overline{|\nabla_h \theta^{(m)}|^2}(z) + \left(\overline{w_m\theta^{(m)}}\right)^2(z) \right] ds \\
&= \frac{1}{2}\overline{|\theta^{(m)}|^2}(z,0) = \frac{1}{2}\overline{\theta_0^2}(z) \leq C\left(\|\theta_0\|_2^2 + \|\partial_z \theta_0\|_2^2 \right), \qquad (3.24)
\end{aligned}
$$

for a.e. $z \in [0,1]$, for all $t \in [0,T]$, $m \geq 2$, where the last inequality is due to Lemma 2.2.

3.2.6. Estimate for $\|\nabla_h \theta^{(m)}\|_2^2$

Recall the regularity of $\theta^{(m)}$ given in (3.8). Thus, we can take the $L^2(\Omega)$ inner product of (3.3) with $-\Delta_h \theta^{(m)}$, and after integrating by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla_h \theta^{(m)}\|_2^2 + \frac{1}{Pe}\|\Delta_h \theta^{(m)}\|_2^2 \\
&\leq \int_\Omega \left| (\mathbf{u}_m \cdot \nabla_h \theta^{(m)}) \Delta_h \theta^{(m)} \right| dxdydz + \int_0^1 \left| \overline{w_m \theta^{(m)}} \right| \left(\int_{[0,L]^2} |\theta^{(m)} \Delta_h w_m| dxdy \right) dz \\
&\leq C \|\nabla_h \mathbf{u}_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h \theta^{(m)}\|_2^{1/2} \|\Delta_h \theta^{(m)}\|_2^{3/2} \\
&\quad + \|\overline{w_m \theta^{(m)}}\|_2 \|\Delta_h w_m\|_2 \|\overline{|\theta^{(m)}|^2}\|_{L^\infty}^{1/2}, \qquad (3.25)
\end{aligned}
$$

where we have used Lemma 2.1, the Cauchy-Schwarz inequality, and Poincaré inequality (1.9).

Employing the Young's inequality implies

$$
\begin{aligned}
\frac{d}{dt}\|\nabla_h \theta^{(m)}\|_2^2 + \frac{1}{Pe}\|\Delta_h \theta^{(m)}\|_2^2 &\leq C\|\nabla_h \mathbf{u}_m\|_2^2 \left(\|\mathbf{u}_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2 \right) \|\nabla_h \theta^{(m)}\|_2^2 \\
&\quad + \|\overline{w_m\theta^{(m)}}\|_2^2 \|\overline{|\theta^{(m)}|^2}\|_{L^\infty} + \|\Delta_h w_m\|_2^2. \qquad (3.26)
\end{aligned}
$$

Applying the Gronwall's inequality to (3.26), we have, for $m \geq 2$,

$$
\begin{aligned}
&\|\nabla_h\theta^{(m)}(t)\|_2^2+\frac{1}{Pe}\int_0^t\|\Delta_h\theta^{(m)}\|_2^2ds\\
&\le\left[\|\nabla_h\theta_0\|_2^2+\int_0^t\left(\|\overline{w_m\theta^{(m)}}\|_2^2\|\overline{|\theta^{(m)}|^2}\|_{L^\infty}+\|\Delta_h w_m\|_2^2\right)ds\right]e^{C\int_0^t\|\nabla_h\mathbf{u}_m\|_2^2(\|\mathbf{u}_m\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2)ds}\\
&\le C(\|\theta_0\|_{H^1},\|\nabla_h w_0\|_2,\|\omega_0\|_2),\quad \text{for } t\in[0,T],
\end{aligned}
\tag{3.27}
$$

due to the bounds (3.11), (3.19), (3.22) and (3.24).

3.2.7. Estimate for $\|\partial_z w_m\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2+\|\partial_z\theta^{(m)}\|_2^2$

We differentiate (3.1)-(3.2) with respect to z and multiply them by $\partial_z w_m$ and $\partial_z\phi_m$ respectively. Integrating the obtained equations over $\Omega\times[0,t]$ yields

$$
\begin{aligned}
&\frac{1}{2}\left(\|\partial_z w_m(t)\|_2^2+\|\partial_z\mathbf{u}_m(t)\|_2^2\right)+\frac{1}{Re}\int_0^t\left(\|\nabla_h\partial_z w_m\|_2^2+\|\partial_z\omega_m\|_2^2\right)ds+\epsilon^2\int_0^t\|\partial_{zz}\phi_m\|_2^2ds\\
&\le\frac{1}{2}\left(\|\partial_z w_m(0)\|_2^2+\|\partial_z\mathbf{u}_m(0)\|_2^2\right)+\int_0^t\int_\Omega|(\partial_z\mathbf{u}_m\cdot\nabla_h w_m)\partial_z w_m|dxdydzds\\
&\quad+\int_0^t\int_\Omega|(\mathbf{u}_m\cdot\nabla_h\partial_z\phi_m)\partial_z\omega_m|dxdydzds+\frac{1}{2}\int_0^t\left(\|\partial_z\theta^{(m-1)}\|_2^2+\Gamma^2\|\partial_z w_m\|_2^2\right)ds,
\end{aligned}
\tag{3.28}
$$

for $t\in[0,T]$, where we have used (1.11) and (1.14).

Now, we estimate each nonlinear term in (3.28). By Lemma 2.1 with $f=\nabla_h w_m$, $g=\partial_z\mathbf{u}_m$ and $h=\partial_z w_m$, we have

$$
\begin{aligned}
&\int_\Omega|(\partial_z\mathbf{u}_m\cdot\nabla_h w_m)\partial_z w_m|dxdydz\\
&\le C\|\Delta_h w_m\|_2^{1/2}(\|\nabla_h w_m\|_2+\|\nabla_h\partial_z w_m\|_2)^{1/2}\|\partial_z\mathbf{u}_m\|_2^{1/2}\|\partial_z\omega_m\|_2^{1/2}\|\partial_z w_m\|_2\\
&\le C\|\Delta_h w_m\|_2\|\partial_z\mathbf{u}_m\|_2^{1/2}\|\partial_z\omega_m\|_2^{1/2}\|\partial_z w_m\|_2\\
&\quad+C\|\Delta_h w_m\|_2^{1/2}\|\nabla_h\partial_z w_m\|_2^{1/2}\|\partial_z\mathbf{u}_m\|_2^{1/2}\|\partial_z\omega_m\|_2^{1/2}\|\partial_z w_m\|_2\\
&\le\frac{1}{4Re}\left(\|\nabla_h\partial_z w_m\|_2^2+\|\partial_z\omega_m\|_2^2\right)+\|\partial_z\mathbf{u}_m\|_2^2+C\left(\|\Delta_h w_m\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2\right)\|\partial_z w_m\|_2^2.
\end{aligned}
\tag{3.29}
$$

Also using Lemma 2.1 with $f=\mathbf{u}_m$, $g=\nabla_h\partial_z\phi_m$ and $h=\partial_z\omega_m$, we obtain

$$
\begin{aligned}
&\int_\Omega|(\mathbf{u}_m\cdot\nabla_h\partial_z\phi_m)\partial_z\omega_m|dxdydz\\
&\le C\|\omega_m\|_2^{1/2}(\|\mathbf{u}_m\|_2+\|\partial_z\mathbf{u}_m\|_2)^{1/2}\|\partial_z\mathbf{u}_m\|_2^{1/2}\|\partial_z\omega_m\|_2^{3/2}
\end{aligned}
$$

$$\leq \frac{1}{4Re}\|\partial_z \omega_m\|_2^2 + C\|\omega_m\|_2^2 \left(\|\mathbf{u}_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2\right) \|\partial_z \mathbf{u}_m\|_2^2. \tag{3.30}$$

Applying (3.29)-(3.30) to the right-hand side of inequality (3.28) yields

$$\begin{aligned}
&\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \frac{1}{Re}\int_0^t \left(\|\nabla_h \partial_z w_m\|_2^2 + \|\partial_z \omega_m\|_2^2\right) ds + \epsilon^2 \int_0^t \|\partial_{zz}\phi_m\|_2^2 ds \\
&\leq \|\partial_z w_m(0)\|_2^2 + \|\partial_z \mathbf{u}_m(0)\|_2^2 + C\int_0^t \left(\|\omega_m\|_2^2\|\mathbf{u}_m\|_2^2 + \|\omega_m\|_2^2\|\partial_z \mathbf{u}_m\|_2^2 + 1\right)\|\partial_z \mathbf{u}_m\|_2^2 ds \\
&\quad + C\int_0^t \left(\|\Delta_h w_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2 + 1\right)\|\partial_z w_m\|_2^2 ds + \int_0^t \|\partial_z \theta^{(m-1)}\|_2^2 ds,
\end{aligned} \tag{3.31}$$

for all $t \in [0, T]$.

Next, we estimate $\|\partial_z \theta^{(m)}\|_2^2$. Since $\theta^{(m)}$ has regularity (3.8)-(3.9) and $\mathbf{u}_m$, w_m are analytic, we can differentiate (3.3) with respect to z, and then multiply it by $\partial_z \theta^{(m)}$, and finally integrate the result with respect to horizontal variables over $[0, L]^2$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{[0,L]^2} |\partial_z \theta^{(m)}|^2 dxdy + \int_{[0,L]^2} (\partial_z \mathbf{u}_m \cdot \nabla_h \theta^{(m)})\partial_z \theta^{(m)} dxdy \\
&\quad + \overline{w_m \theta^{(m)}} \int_{[0,L]^2} (\partial_z w_m)(\partial_z \theta^{(m)}) dxdy + \overline{(\partial_z w_m)\theta^{(m)}} \int_{[0,L]^2} w_m(\partial_z \theta^{(m)}) dxdy \\
&\quad + \overline{w_m(\partial_z \theta^{(m)})} \int_{[0,L]^2} w_m(\partial_z \theta^{(m)}) dxdy \\
&= -\frac{1}{Pe}\int_{[0,L]^2} |\nabla_h \partial_z \theta^{(m)}|^2 dxdy, \quad \text{for a.e. } z \in [0, 1].
\end{aligned}$$

Recall the notation for the horizontal mean $\overline{f} = \frac{1}{L^2}\int_{[0,L]^2} f\,dxdy$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{[0,L]^2} |\partial_z \theta^{(m)}|^2 dxdy + \frac{1}{Pe}\int_{[0,L]^2} |\nabla_h \partial_z \theta^{(m)}|^2 dxdy + L^2 \left|\overline{w_m(\partial_z \theta^{(m)})}\right|^2 \\
&\leq \int_{[0,L]^2} \left|(\partial_z \mathbf{u}_m \cdot \nabla_h \theta^{(m)})\partial_z \theta^{(m)}\right| dxdy + \left|\overline{w_m \theta^{(m)}}\right| \int_{[0,L]^2} |\partial_z w_m||\partial_z \theta^{(m)}| dxdy \\
&\quad + L^2 \left|\overline{(\partial_z w_m)\theta^{(m)}}\right| \left|\overline{w_m(\partial_z \theta^{(m)})}\right|, \quad \text{for a.e. } z \in [0, 1].
\end{aligned} \tag{3.32}$$

Note,

$$L^2\left(\left|\overline{(\partial_z w_m)\theta^{(m)}}\right|\left|\overline{w_m(\partial_z\theta^{(m)})}\right|\right)(z) \le L^2\left|\overline{(\partial_z w_m)\theta^{(m)}}\right|^2(z) + L^2\left|\overline{w_m(\partial_z\theta^{(m)})}\right|^2(z)$$
$$\le \overline{|\theta^{(m)}|^2}(z)\int_{[0,L]^2}|\partial_z w_m(x,y,z)|^2 dxdy + L^2\left|\overline{w_m(\partial_z\theta^{(m)})}\right|^2(z), \quad \text{for a.e. } z\in[0,1].$$

Thus, (3.32) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{[0,L]^2}|\partial_z\theta^{(m)}(x,y,z)|^2 dxdy + \frac{1}{Pe}\int_{[0,L]^2}|\nabla_h\partial_z\theta^{(m)}(x,y,z)|^2 dxdy\\
&\le \int_{[0,L]^2}\left|(\partial_z\mathbf{u}_m\cdot\nabla_h\theta^{(m)})\partial_z\theta^{(m)}\right|(x,y,z)dxdy + \overline{|\theta^{(m)}|^2}(z)\int_{[0,L]^2}|\partial_z w_m(x,y,z)|^2 dxdy\\
&\quad + \left|\overline{w_m\theta^{(m)}}\right|(z)\left(\int_{[0,L]^2}|\partial_z w_m(x,y,z)|^2 dxdy\right)^{1/2}\left(\int_{[0,L]^2}|\partial_z\theta^{(m)}(x,y,z)|^2 dxdy\right)^{1/2},
\end{aligned} \tag{3.33}$$

for a.e. $z\in[0,1]$. Applying Lemma 2.3 to (3.33) with $\eta(z,t)=\left(\int_{[0,L]^2}|\partial_z\theta^{(m)}(x,y,z,t)|^2dxdy\right)^{1/2}$, and using Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
&\int_{[0,L]^2}|\partial_z\theta^{(m)}(x,y,z,t)|^2 dxdy + \frac{2}{Pe}\int_0^t\int_{[0,L]^2}|\nabla_h\partial_z\theta^{(m)}(x,y,z,s)|^2 dxdyds\\
&\le C\int_{[0,L]^2}|\partial_z\theta^{(m)}(x,y,z,0)|^2 dxdy + C\int_0^t\int_{[0,L]^2}\left|(\partial_z\mathbf{u}_m\cdot\nabla_h\theta^{(m)})\partial_z\theta^{(m)}\right|(x,y,z,s)dxdyds\\
&\quad + C\int_0^t\left(\overline{|\theta^{(m)}|^2}(z,s)\int_{[0,L]^2}|\partial_z w_m(x,y,z,s)|^2 dxdy\right)ds\\
&\quad + C\left(\int_0^t\left|\overline{w_m\theta^{(m)}}\right|^2(z,s)ds\right)\int_0^t\int_{[0,L]^2}|\partial_z w_m(x,y,z,s)|^2 dxdyds,
\end{aligned} \tag{3.34}$$

for a.e. $z\in[0,1]$, and for all $t\in[0,T]$.

Then, we integrate (3.34) with respect to z over $[0,1]$ to get

$$\|\partial_z\theta^{(m)}(t)\|_2^2 + \frac{2}{Pe}\int_0^t\|\nabla_h\partial_z\theta^{(m)}\|_2^2 ds$$

$$
\begin{aligned}
&\le C\|\partial_z\theta^m(0)\|_2^2 + C\int_0^t\int_\Omega \left|(\partial_z \mathbf{u}_m\cdot\nabla_h\theta^{(m)})\partial_z\theta^{(m)}\right| dxdydzds\\
&+ C\int_0^t \|\overline{|\theta^{(m)}|^2}\|_{L^\infty}\|\partial_z w_m\|_2^2 ds + C\left(\sup_{z\in[0,1]}\int_0^t \left|\overline{w_m\theta^{(m)}}\right|^2(z,s)ds\right)\int_0^t \|\partial_z w_m\|_2^2 ds.
\end{aligned}
\tag{3.35}
$$

Note that by using Lemma 2.1, with $f=\nabla_h\theta^{(m)}$, $g=\partial_z\mathbf{u}_m$ and $h=\partial_z\theta^{(m)}$, and Poincaré inequality (1.9), one has

$$
\begin{aligned}
&C\int_\Omega \left|(\partial_z \mathbf{u}_m\cdot\nabla_h\theta^{(m)})\partial_z\theta^{(m)}\right| dxdydz\\
&\le C\|\Delta_h\theta^{(m)}\|_2^{1/2}\left(\|\nabla_h\theta^{(m)}\|_2+\|\nabla_h\partial_z\theta^{(m)}\|_2\right)^{1/2}\|\partial_z\mathbf{u}_m\|_2^{1/2}\|\partial_z\omega_m\|_2^{1/2}\|\partial_z\theta^{(m)}\|_2\\
&\le \frac{1}{Pe}\|\nabla_h\partial_z\theta^{(m)}\|_2^2+\frac{1}{2Re}\|\partial_z\omega_m\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2+C\left(\|\Delta_h\theta^{(m)}\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2\right)\|\partial_z\theta^{(m)}\|_2^2.
\end{aligned}
\tag{3.36}
$$

Substituting (3.36) into (3.35), we obtain

$$
\begin{aligned}
&\|\partial_z\theta^{(m)}(t)\|_2^2+\frac{1}{Pe}\int_0^t \|\nabla_h\partial_z\theta^{(m)}\|_2^2 ds\\
&\le C\|\partial_z\theta^{(m)}(0)\|_2^2+\frac{1}{2Re}\int_0^t\|\partial_z\omega_m\|_2^2 ds+\int_0^t\|\partial_z\mathbf{u}_m\|_2^2 ds\\
&\quad + C\int_0^t\left(\|\Delta_h\theta^{(m)}\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2\right)\|\partial_z\theta^{(m)}\|_2^2 ds + C\int_0^t \|\overline{|\theta^{(m)}|^2}\|_{L^\infty}\|\partial_z w_m\|_2^2 ds\\
&\quad + C\left(\sup_{z\in[0,1]}\int_0^t\left|\overline{w_m\theta^{(m)}}\right|^2(z,s)ds\right)\int_0^t\|\partial_z w_m\|_2^2 ds,\quad \text{for all } t\in[0,T].
\end{aligned}
\tag{3.37}
$$

Combining (3.31) and (3.37) provides

$$
\begin{aligned}
&\|\partial_z w_m(t)\|_2^2+\|\partial_z\mathbf{u}_m(t)\|_2^2+\|\partial_z\theta^{(m)}(t)\|_2^2+\frac{1}{2Re}\int_0^t\left(\|\nabla_h\partial_z w_m\|_2^2+\|\partial_z\omega_m\|_2^2\right)ds\\
&+\frac{1}{Pe}\int_0^t\|\nabla_h\partial_z\theta^{(m)}\|_2^2 ds+\epsilon^2\int_0^t\|\partial_{zz}\phi_m\|_2^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\partial_z w_m(0)\|_2^2 + \|\partial_z \mathbf{u}_m(0)\|_2^2 + C\|\partial_z \theta^{(m)}(0)\|_2^2 + C\int_0^t \left(\|\Delta_h w_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2 + 1\right)\|\partial_z w_m\|_2^2 ds \\
&+ C\int_0^t \|\overline{|\theta^{(m)}|^2}\|_{L^\infty}\|\partial_z w_m\|_2^2 ds + C\left(\sup_{z\in[0,1]}\int_0^t \left|\overline{w_m\theta^{(m)}}\right|^2 ds\right)\int_0^t \|\partial_z w_m\|_2^2 ds \\
&+ C\int_0^t \left(\|\omega_m\|_2^2\|\mathbf{u}_m\|_2^2 + \|\omega_m\|_2^2\|\partial_z \mathbf{u}_m\|_2^2 + 1\right)\|\partial_z \mathbf{u}_m\|_2^2 ds \\
&+ C\int_0^t \left(\|\Delta_h \theta^{(m)}\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2\right)\|\partial_z \theta^{(m)}\|_2^2 ds + \int_0^t \|\partial_z \theta^{(m-1)}\|_2^2 ds, \qquad (3.38)
\end{aligned}
$$

for all $t \in [0, T]$.

Thanks to the Gronwall's inequality, we obtain, for all $t \in [0, T]$, $m \geq 2$,

$$
\begin{aligned}
&\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \|\partial_z \theta^{(m)}(t)\|_2^2 + \frac{1}{2Re}\int_0^t \left(\|\nabla_h \partial_z w_m\|_2^2 + \|\partial_z \omega_m\|_2^2\right) ds \\
&+ \frac{1}{Pe}\int_0^t \|\nabla_h \partial_z \theta^{(m)}\|_2^2 ds + \epsilon^2 \int_0^t \|\partial_{zz}\phi_m\|_2^2 ds \\
&\leq \left(\|\partial_z w_m(0)\|_2^2 + \|\partial_z \mathbf{u}_m(0)\|_2^2 + C\|\partial_z \theta^{(m)}(0)\|_2^2 + \int_0^t \|\partial_z \theta^{(m-1)}\|_2^2 ds\right) e^{M(t)}, \qquad (3.39)
\end{aligned}
$$

where

$$
\begin{aligned}
M(t) &= C\int_0^t \Big(\|\Delta_h w_m\|_2^2 + \|\partial_z \mathbf{u}_m\|_2^2 + \|\omega_m\|_2^2\|\mathbf{u}_m\|_2^2 + \|\omega_m\|_2^2\|\partial_z \mathbf{u}_m\|_2^2 \\
&\qquad + \|\Delta_h \theta^{(m)}\|_2^2 + \|\overline{|\theta^{(m)}|^2}\|_{L^\infty} + \sup_{z\in[0,1]}\int_0^t \left|\overline{w_m\theta^{(m)}}\right|^2 d\tau + 1\Big) ds \\
&\leq \tilde{C}(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T), \quad \text{for all } t \in [0, T], \qquad (3.40)
\end{aligned}
$$

owing to estimates (3.15), (3.19), (3.22), (3.24) and (3.27).

In order to show the right-hand side of (3.39) is uniformly bounded, we must prove that $\int_0^T \|\partial_z \theta^{(m)}\|_2^2 ds$ is uniformly bounded. Set $t_0 = 1/(2e^{\tilde{C}})$, where $\tilde{C} > 0$ is given in (3.40). We first show $\int_0^{t_0} \|\partial_z \theta^{(m)}\|_2^2 ds$ is a bounded sequence. Then we show $\int_{t_0}^{2t_0} \|\partial_z \theta^{(m)}\|_2^2 ds$ is a bounded sequence. After iterating finitely many times, we will conclude that the sequence $\int_0^T \|\partial_z \theta^{(m)}\|_2^2 ds$ is bounded.

To see $\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds$ is a bounded sequence, we argue by induction. To begin with, we obtain from (3.7) that

$$\int_0^t \|\partial_z\theta^{(1)}\|_2^2 ds \le C\|\partial_z\theta^{(1)}(0)\|_2^2 = C\|\partial_z\theta_0\|_2^2, \quad \text{for all } t\in[0,T]. \tag{3.41}$$

For an index $m\ge 2$, we consider two cases. If $\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds < \int_0^{t_0} \|\partial_z\theta^{(m-1)}\|_2^2 ds$, then the sequence $\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds$ decreases at the level m. Conversely, if $\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds \ge \int_0^{t_0} \|\partial_z\theta^{(m-1)}\|_2^2 ds$, then by (3.39)-(3.40), we obtain, for $t\in[0,t_0]$,

$$\begin{aligned}
&\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \|\partial_z\theta^{(m)}(t)\|_2^2 \\
&\le \left(\|\partial_z w_m(0)\|_2^2 + \|\partial_z \mathbf{u}_m(0)\|_2^2 + C\|\partial_z\theta^{(m)}(0)\|_2^2 + \int_0^{t_0} \|\partial_z\theta^{(m-1)}\|_2^2 ds \right) e^{M(t)} \\
&\le e^{\tilde C}\left(\|\partial_z w_0\|_2^2 + \|\partial_z \mathbf{u}_0\|_2^2 + C\|\partial_z\theta_0\|_2^2\right) + e^{\tilde C}\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds \\
&\le C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T) + e^{\tilde C}\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds,
\end{aligned} \tag{3.42}$$

where $\tilde C > 0$ is the constant given in (3.40), depending only on $\|\mathbf{u}_0\|_{H^1}$, $\|w_0\|_{H^1}$, $\|\theta_0\|_{H^1}$ and T.

Integrating (3.42) over $[0,t_0]$ provides

$$\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds \le C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T) + t_0\cdot e^{\tilde C}\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds. \tag{3.43}$$

Since $t_0 = 1/(2e^{\tilde C})$, then $t_0\cdot e^{\tilde C} = \frac{1}{2}$. Thus, we obtain from (3.43) that

$$\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds \le C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T). \tag{3.44}$$

Since the right-hand side of (3.44) is independent of m, by induction, we obtain the sequence $\int_0^{t_0} \|\partial_z\theta^{(m)}\|_2^2 ds$ is bounded by $C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T)$. As a result, by (3.39), we have

$$\begin{aligned}
&\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \|\partial_z\theta^{(m)}(t)\|_2^2 \\
&\le C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T), \quad \text{for all } t\in[0,t_0],\ m\ge 2.
\end{aligned} \tag{3.45}$$

We remark that the constant $C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T)$ may vary from line to line in our estimates, but it is always independent of m. On the other hand, $\tilde C$ is a fixed constant given in (3.40), also independent of m.

Next, we show that $\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds$ is a bounded sequence. Indeed, repeating the same estimates as in (3.28)-(3.40) with the time starting at t_0, we have, for all $t\in[t_0,T]$,

$$\begin{aligned}&\|\partial_z w_m(t)\|_2^2+\|\partial_z \mathbf{u}_m(t)\|_2^2+\|\partial_z\theta^{(m)}(t)\|_2^2\\&\le\left(\|\partial_z w_m(t_0)\|_2^2+\|\partial_z \mathbf{u}_m(t_0)\|_2^2+C\|\partial_z\theta^{(m)}(t_0)\|_2^2+\int_{t_0}^{t}\|\partial_z\theta^{(m-1)}\|_2^2 ds\right)e^{M_1(t)},\end{aligned}\tag{3.46}$$

for $m\ge 2$, where

$$\begin{aligned}M_1(t)&=C\int_{t_0}^{t}\big(\|\Delta_h w_m\|_2^2+\|\partial_z\mathbf{u}_m\|_2^2+\|\omega_m\|_2^2\|\mathbf{u}_m\|_2^2+\|\omega_m\|_2^2\|\partial_z\mathbf{u}_m\|_2^2\\&\qquad+\|\Delta_h\theta^{(m)}\|_2^2+\|\overline{|\theta^{(m)}|^2}\|_{L^\infty}+\sup_{z\in[0,1]}\int_{t_0}^{t}\left|\overline{w_m\theta^{(m)}}\right|^2 d\tau+1\big)ds\\&\le M(t)\le\tilde{C}(\|\mathbf{u}_0\|_{H^1},\|w_0\|_{H^1},\|\theta_0\|_{H^1},T),\quad\text{for all } t\in[t_0,T],\end{aligned}\tag{3.47}$$

due to (3.40).

Now, we can see that $\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds$ is a bounded sequence via induction. Indeed, we should first notice that $\int_{t_0}^{2t_0}\|\partial_z\theta^{(1)}\|_2^2 ds\le C\|\partial_z\theta_0\|_2^2$ due to (3.41). Then, for an index $m\ge 2$, if $\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds\ge\int_{t_0}^{2t_0}\|\partial_z\theta^{(m-1)}\|_2^2 ds$, then by (3.46), we obtain, for $t\in[t_0,2t_0]$,

$$\begin{aligned}&\|\partial_z w_m(t)\|_2^2+\|\partial_z \mathbf{u}_m(t)\|_2^2+\|\partial_z\theta^{(m)}(t)\|_2^2\\&\le\left(\|\partial_z w_m(t_0)\|_2^2+\|\partial_z \mathbf{u}_m(t_0)\|_2^2+C\|\partial_z\theta^{(m)}(t_0)\|_2^2+\int_{t_0}^{2t_0}\|\partial_z\theta^{(m-1)}\|_2^2 ds\right)e^{M_1(t)},\\&\le\left(C(\|\mathbf{u}_0\|_{H^1},\|w_0\|_{H^1},\|\theta_0\|_{H^1},T)+\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds\right)e^{\tilde{C}}\\&\le C(\|\mathbf{u}_0\|_{H^1},\|w_0\|_{H^1},\|\theta_0\|_{H^1},T)+e^{\tilde{C}}\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds,\end{aligned}\tag{3.48}$$

where we have used (3.45) and (3.47). Then integrating (3.48) over $[t_0,2t_0]$ and using $t_0\cdot e^{\tilde{C}}=\frac{1}{2}$, we have

$$\int_{t_0}^{2t_0}\|\partial_z\theta^{(m)}\|_2^2 ds\le C(\|\mathbf{u}_0\|_{H^1},\|w_0\|_{H^1},\|\theta_0\|_{H^1},T).\tag{3.49}$$

Hence, by induction, we see that $\int_{t_0}^{2t_0} \|\partial_z \theta^{(m)}\|_2^2 ds$ is bounded by $C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T)$ for all $m \geq 2$. Thus, by (3.46), we obtain, for all $t \in [t_0, 2t_0]$,

$$\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \|\partial_z \theta^{(m)}(t)\|_2^2 \leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T).$$

After iterating the above procedure on finitely many intervals $[0, t_0]$, $[t_0, 2t_0]$, $\cdots$, $[nt_0, T]$, we eventually obtain $\int_0^T \|\partial_z \theta^{(m)}\|_2^2 dt$ is a bounded sequence, namely,

$$\int_0^T \|\partial_z \theta^{(m)}\|_2^2 dt \leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T), \quad \text{for any } m \in \mathbb{N}. \tag{3.50}$$

By substituting (3.50) to the right-hand side of (3.39), we achieve the desired uniform bound

$$\begin{aligned}
&\|\partial_z w_m(t)\|_2^2 + \|\partial_z \mathbf{u}_m(t)\|_2^2 + \|\partial_z \theta^{(m)}(t)\|_2^2 + \frac{1}{2Re}\int_0^t \left(\|\nabla_h \partial_z w_m\|_2^2 + \|\partial_z \omega_m\|_2^2\right) ds \\
&\quad + \frac{1}{Pe}\int_0^t \|\nabla_h \partial_z \theta^{(m)}\|_2^2 ds + \epsilon^2 \int_0^t \|\partial_{zz}\phi_m\|_2^2 ds \\
&\leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{H^1}, T), \quad \text{for all } t \in [0, T],\ m \geq 2.
\end{aligned} \tag{3.51}$$

3.3. Passage to the limit

According to all of the estimates which have been established in section 3.2 for $\mathbf{u}_m$, ω_m, w_m and $\theta^{(m)}$, we obtain the following uniform bounds:

$$\mathbf{u}_m,\ w_m,\ \theta^{(m)} \text{ are uniformly bounded in } L^\infty(0, T; H^1(\Omega)); \tag{3.52}$$

$$\omega_m \text{ is uniformly bounded in } L^\infty(0, T; L^2(\Omega)); \tag{3.53}$$

$$\nabla_h \omega_m,\ \Delta_h w_m,\ \Delta_h \theta^{(m)},\ \overline{w_m \theta^{(m)}} \text{ are uniformly bounded in } L^2(\Omega \times (0, T)); \tag{3.54}$$

$$\partial_z \omega_m,\ \nabla_h \partial_z w_m,\ \nabla_h \partial_z \theta^{(m)},\ \partial_{zz}\phi_m \text{ are uniformly bounded in } L^2(\Omega \times (0, T)). \tag{3.55}$$

Therefore, on a subsequence, as $m \to \infty$,

$$\mathbf{u}_m \to \mathbf{u},\ w_m \to w,\ \theta^{(m)} \to \theta \text{ weakly* in } L^\infty(0, T; H^1(\Omega)); \tag{3.56}$$

$$\omega_m \to \omega \text{ weakly* in } L^\infty(0, T; L^2(\Omega)); \tag{3.57}$$

$$\nabla_h \omega_m \to \nabla_h \omega,\ \Delta_h w_m \to \Delta_h w,\ \Delta_h \theta^{(m)} \to \Delta_h \theta \text{ weakly in } L^2(\Omega \times (0, T)); \tag{3.58}$$

$$\partial_z \omega_m \to \omega_z,\ \nabla_h \partial_z w_m \to \nabla_h w_z,\ \nabla_h \partial_z \theta^{(m)} \to \nabla_h \theta_z,\ \partial_{zz}\phi_m \to \phi_{zz} \text{ weakly in } L^2(\Omega \times (0, T)). \tag{3.59}$$

In order to use a compactness theorem to obtain certain strong convergence of the approximate solutions, we shall derive uniform bounds independent of $m \geq 2$ for $\partial_t w_m$, $\partial_t \mathbf{u}_m$, $\partial_t \omega_m$ and

$\partial_t \theta^{(m)}$. First, we claim that the sequence $\partial_t w_m$ is uniformly bounded in $L^2(\Omega \times (0,T))$. Indeed, for any function $\eta \in L^{4/3}(0,T;L^2(\Omega))$, we use Lemma 2.1 to estimate

$$
\begin{aligned}
&\int_0^T \int_\Omega (\mathbf{u}_m \cdot \nabla_h w_m)\eta \, dxdydzdt \\
&\le C \int_0^T \|\omega_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h w_m\|_2^{1/2} \|\Delta_h w_m\|_2^{1/2} \|\eta\|_2 dt \\
&\le C \sup_{t\in[0,T]} \Big[\|\omega_m\|_2^{\frac{1}{2}} (\|\mathbf{u}_m\|_2^{\frac{1}{2}} + \|\partial_z \mathbf{u}_m\|_2^{\frac{1}{2}}) \|\nabla_h w_m\|_2^{\frac{1}{2}}\Big] \Big(\int_0^T \|\Delta_h w_m\|_2^2 dt\Big)^{\frac{1}{4}} \Big(\int_0^T \|\eta\|_2^{\frac{4}{3}} dt\Big)^{\frac{3}{4}},
\end{aligned}
\tag{3.60}
$$

which is uniformly bounded due to (3.52) and (3.54). Consequently, the sequence $\mathbf{u}_m \cdot \nabla_h w_m$ is bounded in $L^4(0,T;L^2(\Omega))$. As a result, we obtain from equation (3.1) that

$$
\partial_t w_m \text{ is uniformly bounded in } L^2(\Omega \times (0,T)). \tag{3.61}
$$

Next we show that $\partial_t \mathbf{u}_m$ is bounded in $L^2(\Omega \times (0,T))$. For any function $\tilde{\eta} \in L^2(0,T;H_h^1(\Omega))$, we apply Lemma 2.1 and Poincaré inequality (1.9) to estimate

$$
\begin{aligned}
&\int_0^T \int_\Omega (\mathbf{u}_m \cdot \nabla_h \omega_m)\tilde{\eta} \, dxdydzdt \\
&\le C \int_0^T \|\nabla_h \mathbf{u}_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h \omega_m\|_2 \|\tilde{\eta}\|_2^{1/2} (\|\tilde{\eta}\|_2 + \|\nabla_h \tilde{\eta}\|_2)^{1/2} dt \\
&\le C \sup_{t\in[0,T]} \Big[\|\nabla_h \mathbf{u}_m\|_2^{\frac{1}{2}} (\|\mathbf{u}_m\|_2^{\frac{1}{2}} + \|\partial_z \mathbf{u}_m\|_2^{\frac{1}{2}})\Big] \Big(\int_0^T \|\nabla_h \omega_m\|_2^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T \|\tilde{\eta}\|_2^2 + \|\nabla_h \tilde{\eta}\|_2^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.62}
$$

Note that (3.52)-(3.54) provide the uniform bound for the right-hand side of (3.62). Therefore, the sequence $\mathbf{u}_m \cdot \nabla_h \omega_m$ is bounded uniformly in m in $L^2(0,T;(H_h^1(\Omega))')$, where $(H_h^1(\Omega))'$ is the dual space of $H_h^1(\Omega)$. Consequently, we obtain from the vorticity equation (3.2) that

$$
\partial_t \omega_m \text{ is uniformly bounded in } L^2(0,T;(H_h^1(\Omega))'); \tag{3.63}
$$

$$
\partial_t \mathbf{u}_m \text{ is uniformly bounded in } L^2(\Omega \times (0,T)). \tag{3.64}
$$

Moreover, $\partial_t \theta^{(m)}$ is bounded in $L^2(\Omega \times (0,T))$. Indeed, applying Hölder's inequality, we deduce

$$\int_\Omega |w_m\overline{w_m\theta^{(m)}}|^2dxdydz \le C\int_0^1\left(\int_{[0,L]^2}|w_m|^2dxdy\right)^2\left(\int_{[0,L]^2}|\theta^{(m)}|^2dxdy\right)dz$$
$$\le C\int_0^1\left(\int_{[0,L]^2}|w_m|^6dxdy\right)^{2/3}\left(\int_{[0,L]^2}|\theta^{(m)}|^6dxdy\right)^{1/3}dz$$
$$\le C\|w_m\|_6^4\|\theta^{(m)}\|_6^2 \le C\|w_m\|_{H^1}^4\|\theta^{(m)}\|_{H^1}^2, \tag{3.65}$$

where the last inequality is due to the imbedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ in three dimensions. Since the H^1 norms of w_m and $\theta^{(m)}$ are uniformly bounded on $[0,T]$, (3.65) implies the sequence $w_m\overline{w_m\theta^{(m)}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Also, using an estimate similar to (3.60), we can show the sequence $\mathbf{u}_m\cdot\nabla_h\theta^{(m)}$ is bounded in $L^4(0,T;L^2(\Omega))$. Therefore, we obtain from the temperature equation (1.7) that

$$\partial_t\theta^{(m)} \text{ is uniformly bounded in } L^2(\Omega\times(0,T)). \tag{3.66}$$

Owing to (3.61), (3.63)-(3.64) and (3.66), on a subsequence,

$$\partial_t w_m \to \partial_t w,\ \ \partial_t\mathbf{u}_m \to \partial_t\mathbf{u},\ \ \partial_t\theta^{(m)} \to \partial_t\theta \ \text{ weakly in } L^2(\Omega\times(0,T)); \tag{3.67}$$
$$\partial_t\omega_m \to \partial_t\omega \ \text{ weakly* in } L^2(0,T;(H_h^1(\Omega))'). \tag{3.68}$$

By (3.52), (3.61), (3.64), (3.66), and thanks to the Aubin's compactness theorem, the following strong convergence holds for a subsequence of $(\mathbf{u}_m, w_m, \theta^{(m)})^{tr}$:

$$\mathbf{u}_m \to \mathbf{u},\ \ w_m \to w,\ \ \theta^{(m)} \to \theta \ \text{ in } L^2(\Omega\times(0,T)). \tag{3.69}$$

Thus, $\omega_m \to \omega$ in $L^2(0,T;(H_h^1(\Omega))')$ for this subsequence.

Now we can pass to the limit as $m\to\infty$ for the nonlinear terms in the Galerkin-like system (3.1)-(3.5). Let ψ be a trigonometric polynomial with continuous coefficients. For m larger than the degree of ψ, we have

$$\int_0^T\int_\Omega P_m(\mathbf{u}_m\cdot\nabla_h\omega_m)\psi\,dxdydzdt$$
$$=\int_0^T\int_\Omega(\mathbf{u}\cdot\nabla_h\omega_m)\psi\,dxdydzdt+\int_0^T\int_\Omega[(\mathbf{u}_m-\mathbf{u})\cdot\nabla_h\omega_m]\psi\,dxdydzdt. \tag{3.70}$$

Since $\nabla_h\omega_m \to \nabla_h\omega$ weakly in $L^2(\Omega\times(0,T))$, $\mathbf{u}_m\to\mathbf{u}$ in $L^2(\Omega\times(0,T))$, and $\nabla_h\omega_m$ is bounded in $L^2(\Omega\times(0,T))$, we can pass to the limit in (3.70) to get

$$\lim_{m\to\infty}\int_0^T\int_\Omega P_m(\mathbf{u}_m\cdot\nabla_h\omega_m)\psi\,dxdydzdt=\int_0^T\int_\Omega(\mathbf{u}\cdot\nabla_h\omega)\psi\,dxdydzdt. \tag{3.71}$$

Similarly, we can deduce

$$\lim_{m\to\infty}\int_0^T\int_\Omega P_m(\mathbf{u}_m\cdot\nabla_h w_m)\psi\,dxdydzdt=\int_0^T\int_\Omega(\mathbf{u}\cdot\nabla_h w)\psi\,dxdydzdt. \tag{3.72}$$

$$\lim_{m\to\infty}\int_0^T\int_\Omega(\mathbf{u}_m\cdot\nabla_h\theta^{(m)})\psi\,dxdydzdt=\int_0^T\int_\Omega(\mathbf{u}\cdot\nabla_h\theta)\psi\,dxdydzdt. \tag{3.73}$$

Furthermore, we consider

$$\begin{aligned}&\int_0^T\int_\Omega\left(w_m\overline{w_m\theta^{(m)}}-w\overline{w\theta}\right)\psi\,dxdydzdt\\&=\int_0^T\int_\Omega(w_m-w)\overline{w_m\theta^{(m)}}\psi\,dxdydzdt+\int_0^T\int_\Omega w\overline{(w_m-w)\theta^{(m)}}\psi\,dxdydzdt\\&\quad+\int_0^T\int_\Omega w\overline{w(\theta^{(m)}-\theta)}\psi\,dxdydzdt.\end{aligned} \tag{3.74}$$

We shall show that each integral on the right-hand side of (3.74) converges to zero. The convergence to zero for the first integral on the right-hand side of (3.74) is due to the fact that $w_m\to w$ in $L^2(\Omega\times(0,T))$ and the uniform boundedness of the sequence $\overline{w_m\theta^{(m)}}$ in $L^2(\Omega\times(0,T))$. For the second integral on the right-hand side of (3.74), we apply Cauchy-Schwarz inequality and Lemma 2.2 to get

$$\begin{aligned}&\left|\int_0^T\int_\Omega w\overline{(w_m-w)\theta^{(m)}}\psi\,dxdydzdt\right|\\&\le C\|\psi\|_{L^\infty(\Omega\times(0,T))}\int_0^T\int_0^1\Big(\int_{[0,L]^2}|\theta^{(m)}|^2dxdy\Big)^{\frac12}\Big(\int_{[0,L]^2}|w_m-w|^2dxdy\Big)^{\frac12}\Big(\int_{[0,L]^2}|w|dxdy\Big)dzdt\\&\le C\|\psi\|_{L^\infty(\Omega\times(0,T))}\sup_{t\in[0,T]}\left(\|\theta^{(m)}\|_2+\|\partial_z\theta^{(m)}\|_2\right)\|w_m-w\|_{L^2(\Omega\times(0,T))}\|w\|_{L^2(\Omega\times(0,T))}\longrightarrow 0,\end{aligned}$$

where the convergence to zero is due to the fact that $w_m\to w$ in $L^2(\Omega\times(0,T))$ and the sequence $\theta^{(m)}$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$. Next, we look at the last term on the right-hand side of (3.74):

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega w\overline{w(\theta^{(m)}-\theta)}\psi\,dxdydzdt\right| \\
&\le C\|\psi\|_{L^\infty(\Omega\times(0,T))}\int_0^T\int_0^1\left(\int_{[0,L]^2} w^4dxdy\right)^{1/2}\left(\int_{[0,L]}|\theta^{(m)}-\theta|^2dxdy\right)^{1/2}dzdt \\
&\le C\|\psi\|_{L^\infty(\Omega\times(0,T))}\sup_{t\in[0,T]}\|w\|_4^2\|\theta^{(m)}-\theta\|_{L^2(\Omega\times(0,T))}\longrightarrow 0,
\end{aligned}
$$

where the convergence to zero is due to the fact that $\theta^{(m)}\to\theta$ in $L^2(\Omega\times(0,T))$ and that $w\in L^\infty([0,T];H^1(\Omega))$. In sum, all integrals on the right-hand side of (3.74) converge to zero, and thus

$$
\lim_{m\to\infty}\int_0^T\int_\Omega (w_m\overline{w_m\theta^{(m)}})\psi\,dxdydzdt=\int_0^T\int_\Omega w\overline{w\theta}\psi\,dxdydzdt. \tag{3.75}
$$

Owing to (3.56)-(3.59), (3.67)-(3.68), (3.71)-(3.73), (3.75), we can pass to the limit as $m\to\infty$ for the Galerkin-like system (3.1)-(3.5) to get

$$
\begin{cases}
\int_0^T\int_\Omega\left(\partial_t w+\mathbf{u}\cdot\nabla_h w-\partial_z\phi-\Gamma\theta-\frac{1}{Re}\Delta_h w\right)\psi\,dxdydzdt=0,\\
\int_0^T\int_\Omega\left(\partial_t\omega+\mathbf{u}\cdot\nabla_h\omega-\partial_z w-\frac{1}{Re}\Delta_h\omega-\epsilon^2\partial_{zz}\phi\right)\psi\,dxdydzdt=0,\\
\int_0^T\int_\Omega\left(\partial_t\theta+\mathbf{u}\cdot\nabla_h\theta+w\overline{w\theta}-\frac{1}{Pe}\Delta_h\theta\right)\psi\,dxdydzdt=0,
\end{cases} \tag{3.76}
$$

such that $\nabla_h\cdot\mathbf{u}=0$, for any trigonometric polynomial ψ with continuous coefficients.

By using estimates similar to (3.60) and (3.62), one has $\mathbf{u}\cdot\nabla_h w$, $\mathbf{u}\cdot\nabla_h\theta\in L^4(0,T;L^2(\Omega))$ and $\mathbf{u}\cdot\nabla_h\omega\in L^2(0,T;(H_h^1(\Omega))')$. Also, $w\overline{w\theta}\in L^\infty(0,T;L^2(\Omega))$ due to an estimate like (3.65). Hence, we obtain from (3.76) that equations (1.5)-(1.7) hold in the sense of (1.17). By simply integrating (1.17) in time, we see that $w,\mathbf{u},\theta\in C([0,T];L^2(\Omega))$ and $\omega\in C([0,T];(H_h^1(\Omega))')$. Then, it is easy to verify the initial condition. Also, by (3.10) and (3.69), we find that $\overline{\mathbf{u}}=0$, $\overline{w}=0$ and $\overline{\theta}=0$ for all $t\in[0,T]$. Finally, due to the regularity of solutions, we can multiply (1.17) by $(w,\phi,\theta)^{tr}$ and integrate the result over $\Omega\times[0,t]$ for $t\in[0,T]$ to obtain the energy identity (1.18). This completes the proof for the global existence of strong solutions for system (1.5)-(1.8).

Remark 3.2. If replacing the weak dissipation term $\epsilon^2\frac{\partial^2\phi}{\partial z^2}$ in system (1.5)-(1.8) by the vertical viscosity $\epsilon^2\frac{\partial^2\omega}{\partial z^2}$, then the analysis will be simpler. Indeed, with the help of the vertical viscosity $\epsilon^2\frac{\partial^2\omega}{\partial z^2}$, at the level of the L^2 estimate of the velocity field $(\mathbf{u}_m,w_m)$ for the Galerkin approximation, like the estimate in subsection 3.2.2, one can easily obtain that $\mathbf{u}_m$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$, which will greatly simplify the proof for the existence of global strong solutions.

4. Uniqueness of strong solutions and continuous dependence on initial data

This section is devoted to proving that the strong solutions for system (1.5)-(1.8) are unique and depend continuously on initial data. Since a strong solution has the regularity specified in (1.16), all calculations in this section are legitimate. Assume there are two strong solutions $(\mathbf{u}_1, w_1, \theta_1)^{tr}$ and $(\mathbf{u}_2, w_2, \theta_2)^{tr}$ for system (1.5)-(1.8). Let $\omega_1 = \nabla_h \times \mathbf{u}_1$ and $\omega_2 = \nabla_h \times \mathbf{u}_2$. Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\omega = \omega_1 - \omega_2$, $w = w_1 - w_2$, $\theta = \theta_1 - \theta_2$. Therefore, for a.e. $t \in [0, T]$,

$$\begin{cases} w_t + \mathbf{u} \cdot \nabla_h w_1 + \mathbf{u}_2 \cdot \nabla_h w - \phi_z = \Gamma\theta + \frac{1}{Re}\Delta_h w, \ \text{ in } L^2(\Omega), \\ \omega_t + \mathbf{u} \cdot \nabla_h \omega_1 + \mathbf{u}_2 \cdot \nabla_h \omega - w_z = \frac{1}{Re}\Delta_h \omega + \epsilon^2 \phi_{zz}, \ \text{ in } (H_h^1(\Omega))', \\ \theta_t + \mathbf{u} \cdot \nabla_h \theta_1 + \mathbf{u}_2 \cdot \nabla_h \theta + w\overline{w_1\theta_1} + w_2\overline{w\theta_1} + w_2\overline{w_2\theta} = \frac{1}{Pe}\Delta_h \theta, \ \text{ in } L^2(\Omega), \end{cases} \tag{4.1}$$

and $\nabla_h \cdot \mathbf{u} = 0$.

We multiply (4.1) by $(w, \phi, \theta)^{tr}$ and integrate over Ω. By using (1.10), (1.11), (1.13) and (1.14), we obtain, for a.e. $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2 + \|\mathbf{u}\|_2^2 + \|\theta\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h \mathbf{u}\|_2^2\right) + \frac{1}{Pe}\|\nabla_h\theta\|_2^2 + \epsilon^2\|\phi_z\|_2^2 + \|\overline{w_2\theta}\|_2^2 \\ &\le \int_\Omega |(\mathbf{u}\cdot\nabla_h w)w_1|\,dxdydz + \int_\Omega |(\mathbf{u}_2\cdot\nabla_h\phi)\omega|\,dxdydz + \int_\Omega |(\mathbf{u}\cdot\nabla_h\theta)\theta_1|\,dxdydz \\ &\quad + \frac{\Gamma}{2}\left(\|\theta\|_2^2 + \|w\|_2^2\right) + \int_\Omega |w\overline{w_1\theta_1}\theta|\,dxdydz - \int_\Omega w_2\overline{w\theta_1}\theta\,dxdydz. \end{aligned} \tag{4.2}$$

Now we estimate each term on the right-hand side of (4.2).

Using Lemma 2.1 with $f = w_1$, $g = \mathbf{u}$ and $h = \nabla_h w$, and by Poincaré inequality (1.9), we obtain

$$\begin{aligned} &\int_\Omega |(\mathbf{u}\cdot\nabla_h w)w_1|\,dxdydz \\ &\le C\|\nabla_h w_1\|_2^{1/2}\left(\|w_1\|_2 + \|\partial_z w_1\|_2\right)^{1/2}\|\mathbf{u}\|_2^{1/2}\|\nabla_h\mathbf{u}\|_2^{1/2}\|\nabla_h w\|_2 \\ &\le \frac{1}{6Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + C\|\nabla_h w_1\|_2^2\left(\|w_1\|_2^2 + \|\partial_z w_1\|_2^2\right)\|\mathbf{u}\|_2^2. \end{aligned} \tag{4.3}$$

Also, using Lemma 2.1 with $f = \mathbf{u}_2$, $g = \nabla_h\phi$, $h = \omega$, and by Poincaré inequality (1.9), we have

$$\begin{aligned} \int_\Omega |(\mathbf{u}_2\cdot\nabla_h\phi)\omega|\,dxdydz &\le C\|\omega_2\|_2^{1/2}(\|\mathbf{u}_2\|_2 + \|\partial_z\mathbf{u}_2\|_2)^{1/2}\|\mathbf{u}\|_2^{1/2}\|\nabla_h\mathbf{u}\|_2^{3/2} \\ &\le \frac{1}{6Re}\|\nabla_h\mathbf{u}\|_2^2 + C\|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2 + \|\partial_z\mathbf{u}_2\|_2^2)\|\mathbf{u}\|_2^2. \end{aligned} \tag{4.4}$$

Moreover, by Lemma 2.1 with $f = \theta_1$, $g = \mathbf{u}$, $h = \nabla_h\theta$, and Poincaré inequality (1.9), one has

$$\begin{aligned}&\int_\Omega |(\mathbf{u}\cdot\nabla_h\theta)\theta_1|dxdydz\\&\le \|\nabla_h\theta_1\|_2^{1/2}(\|\theta_1\|_2+\|\partial_z\theta_1\|_2)^{1/2}\|\mathbf{u}\|_2^{1/2}\|\nabla_h\mathbf{u}\|_2^{1/2}\|\nabla_h\theta\|_2\\&\le \frac{1}{6Re}\|\nabla_h\mathbf{u}\|_2^2+\frac{1}{2Pe}\|\nabla_h\theta\|_2^2+C\|\nabla_h\theta_1\|_2^2\left(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2\right)\|\mathbf{u}\|_2^2.\end{aligned}\tag{4.5}$$

Using Cauchy-Schwarz inequality and Lemma 2.2, we get

$$\begin{aligned}\int_\Omega |w\overline{w_1\theta_1}\theta|dxdydz&\le C\|w\|_2\|\theta\|_2\sup_{z\in[0,1]}\left[\Big(\int_{[0,L]^2}w_1^2dxdy\Big)^{1/2}\Big(\int_{[0,L]^2}\theta_1^2dxdy\Big)^{1/2}\right]\\&\le C\|w\|_2\|\theta\|_2(\|w_1\|_2+\|\partial_zw_1\|_2)(\|\theta_1\|_2+\|\partial_z\theta_1\|_2)\\&\le C(\|w_1\|_2^2+\|\partial_zw_1\|_2^2)\|w\|_2^2+(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2)\|\theta\|_2^2.\end{aligned}\tag{4.6}$$

Again, applying Cauchy-Schwarz inequality and Lemma 2.2, we obtain

$$\begin{aligned}-\int_\Omega w_2\overline{w\theta_1}\theta dxdydz&\le\int_0^1\Big(\int_{[0,L]^2}w^2dxdy\Big)^{1/2}\Big(\int_{[0,L]^2}\theta_1^2dxdy\Big)^{1/2}\left|\overline{w_2\theta}\right|dz\\&\le C(\|\theta_1\|_2+\|\partial_z\theta_1\|_2)\|w\|_2\|\overline{w_2\theta}\|_2\\&\le\|\overline{w_2\theta}\|_2^2+C(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2)\|w\|_2^2.\end{aligned}\tag{4.7}$$

Now, we combine estimates (4.2)-(4.7) to deduce, for a.e. $t\in[0,T]$,

$$\begin{aligned}&\frac{d}{dt}\left(\|w\|_2^2+\|\mathbf{u}\|_2^2+\|\theta\|_2^2\right)+\frac{1}{Re}\left(\|\nabla_hw\|_2^2+\|\nabla_h\mathbf{u}\|_2^2\right)+\frac{1}{Pe}\|\nabla_h\theta\|_2^2+\epsilon^2\|\phi_z\|_2^2\\&\le C\Big[\|\nabla_hw_1\|_2^2\left(\|w_1\|_2^2+\|\partial_zw_1\|_2^2\right)+\|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2+\|\partial_z\mathbf{u}_2\|_2^2)\\&\quad+\|\nabla_h\theta_1\|_2^2\left(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2\right)\Big]\|\mathbf{u}\|_2^2\\&\quad+C\left(\|w_1\|_2^2+\|\partial_zw_1\|_2^2+\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2+1\right)\|w\|_2^2+(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2+1)\|\theta\|_2^2.\end{aligned}$$

Thanks to the Gronwall's inequality, we have, for all $t\in[0,T]$,

$$\|w(t)\|_2^2+\|\mathbf{u}(t)\|_2^2+\|\theta(t)\|_2^2\le\left(\|w(0)\|_2^2+\|\mathbf{u}(0)\|_2^2+\|\theta(0)\|_2^2\right)e^{K(t)}\tag{4.8}$$

where

$$\begin{aligned}K(t)=C\int_0^t\Big[&\left(\|\nabla_hw_1\|_2^2+1\right)\left(\|w_1\|_2^2+\|\partial_zw_1\|_2^2\right)+\|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2+\|\partial_z\mathbf{u}_2\|_2^2)\\&+(\|\nabla_h\theta_1\|_2^2+1)\left(\|\theta_1\|_2^2+\|\partial_z\theta_1\|_2^2\right)+1\Big]ds.\end{aligned}$$

Since strong solutions $(\mathbf{u}_1, w_1, \theta_1)^{tr}$ and $(\mathbf{u}_2, w_2, \theta_2)^{tr}$ are in the space $L^\infty(0, T; (H^1(\Omega))^4)$, $K(t)$ is bounded on $[0, T]$. Therefore, (4.8) implies the uniqueness of strong solutions. Furthermore, (4.8) also implies the continuous dependence on initial data for strong solutions, namely, if $\{(\mathbf{u}_0^n, w_0^n, \theta_0^n)^{tr}\}$ is a bounded sequence of initial data in $H^1(\Omega)$ such that $(\mathbf{u}_0^n, w_0^n, \theta_0^n)^{tr} \to (\mathbf{u}_0, w_0, \theta_0)^{tr}$ with respect to the $L^2(\Omega)$ norm, then the corresponding strong solutions $(\mathbf{u}^n, w^n, \theta^n)^{tr} \to (\mathbf{u}, w, \theta)^{tr}$ in $C([0, T]; (L^2(\Omega))^4)$.

5. Large-time behavior

In this section, we prove Theorem 1.3: the asymptotic behavior of strong solutions as $t \to \infty$. Since a strong solution has the regularity specified in (1.16), all calculations in this section are legitimate.

First we show the exponential decay estimates (1.19)-(1.21). Taking the inner product of (1.7) with θ gives

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \frac{1}{Pe}\|\nabla_h\theta\|_2^2 + \|\overline{w\theta}\|_2^2 = 0. \tag{5.1}$$

According to the Poincaré inequality $\|\theta\|_2^2 \le \gamma\|\nabla_h\theta\|_2^2$ and estimate (5.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \frac{1}{\gamma Pe}\|\theta\|_2^2 \le 0. \tag{5.2}$$

It follows that

$$\|\theta(t)\|_2^2 \le e^{-\frac{2}{\gamma Pe}t}\|\theta_0\|_2^2, \quad \text{for all } t \ge 0. \tag{5.3}$$

Then, integrating (5.1) over $[t, t+1]$ gives

$$\int_t^{t+1}\left(\frac{1}{Pe}\|\nabla_h\theta\|_2^2 + \|\overline{w\theta}\|_2^2\right) ds \le \frac{1}{2}\|\theta(t)\|_2^2 \le \frac{1}{2}e^{-\frac{2}{\gamma Pe}t}\|\theta_0\|_2^2, \quad \text{for all } t \ge 0. \tag{5.4}$$

Taking the $L^2(\Omega)$ inner product of equations (1.5)-(1.6) with $(w, \phi)^{tr}$, we deduce

$$\frac{d}{dt}\left(\|w\|_2^2 + \|\mathbf{u}\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + \epsilon^2\|\partial_z\phi\|_2^2 \le C\|\theta\|_2^2. \tag{5.5}$$

Integrating (5.5) over $[0, t]$ gives

$$\begin{aligned} &\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2 + \int_0^t \left(\frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + \epsilon^2\|\partial_z\phi\|_2^2\right) ds \\ &\le \|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 + C\|\theta_0\|_2^2, \quad \text{for all } t \ge 0, \end{aligned} \tag{5.6}$$

by virtue of (5.3).

Applying the Poincaré inequality to (5.5), we have

$$\frac{d}{dt}\left(\|w\|_2^2+\|\mathbf{u}\|_2^2\right)+\frac{1}{\kappa\gamma Re}\left(\|w\|_2^2+\|\mathbf{u}\|_2^2\right)\le C\|\theta\|_2^2, \tag{5.7}$$

for any $\kappa \geq 1$. From (5.7) and (5.3), we obtain

$$\frac{d}{dt}\left(e^{\frac{1}{\kappa\gamma Re}t}\left(\|w\|_2^2+\|\mathbf{u}\|_2^2\right)\right)\le Ce^{\frac{1}{\kappa\gamma Re}t}\|\theta\|_2^2\le Ce^{\left(\frac{1}{\kappa\gamma Re}-\frac{2}{\gamma Pe}\right)t}\|\theta_0\|_2^2. \tag{5.8}$$

We can choose $\kappa \geq 1$ such that $Pe \neq 2\kappa Re$. Then integrating (5.8) over $[0, t]$ implies

$$\|w(t)\|_2^2+\|\mathbf{u}(t)\|_2^2\le e^{-\frac{1}{\kappa\gamma Re}t}\left(\|w_0\|_2^2+\|\mathbf{u}_0\|_2^2\right)+C\left(e^{-\frac{2}{\gamma Pe}t}+e^{-\frac{1}{\kappa\gamma Re}t}\right)\|\theta_0\|_2^2, \tag{5.9}$$

for all $t \geq 0$. Integrating (5.5) from t to $t+1$ shows

$$\begin{aligned}&\int_t^{t+1}\left(\frac{1}{Re}\left(\|\nabla_h w\|_2^2+\|\nabla_h\mathbf{u}\|_2^2\right)+\epsilon^2\|\phi_z\|_2^2\right)ds\le\|w(t)\|_2^2+\|\mathbf{u}(t)\|_2^2+C\int_t^{t+1}\|\theta\|_2^2ds\\ &\le e^{-\frac{1}{\kappa\gamma Re}t}\left(\|w_0\|_2^2+\|\mathbf{u}_0\|_2^2\right)+C\left(e^{-\frac{2}{\gamma Pe}t}+e^{-\frac{1}{\kappa\gamma Re}t}\right)\|\theta_0\|_2^2,\quad \text{for all } t\ge 0,\end{aligned} \tag{5.10}$$

where the last inequality is due to (5.3) and (5.9).

Next, we take the inner product of (1.6) with ω, and adopt the same calculation as in (3.16)-(3.18) to derive

$$\frac{d}{dt}\|\omega\|_2^2+\frac{2}{Re}\|\nabla_h\omega\|_2^2+\epsilon^2\|\mathbf{u}_z\|_2^2\le\frac{1}{\epsilon^2}\|\nabla_h w\|_2^2. \tag{5.11}$$

Integrating (5.11) over $[0, t]$ and using (5.6), we obtain

$$\|\omega(t)\|_2^2+\int_0^t\left(\frac{2}{Re}\|\nabla_h\omega\|_2^2+\epsilon^2\|\mathbf{u}_z\|_2^2\right)ds\le\|\omega_0\|_2^2+C(\|w_0\|_2^2+\|\mathbf{u}_0\|_2^2+\|\theta_0\|_2^2), \tag{5.12}$$

for all $t \geq 0$.

Now, we integrate (5.11) over $[s, t+1]$ for $t \leq s \leq t+1$, and then integrate the result from t to $t+1$ to get

$$\begin{aligned}\|\omega(t+1)\|_2^2&\le\int_t^{t+1}\|\omega\|_2^2ds+\frac{1}{\epsilon^2}\int_t^{t+1}\|\nabla_h w\|_2^2ds\\ &\le Ce^{-\frac{1}{\kappa\gamma Re}t}\left(\|w_0\|_2^2+\|\mathbf{u}_0\|_2^2\right)+C\left(e^{-\frac{2}{\gamma Pe}t}+e^{-\frac{1}{\kappa\gamma Re}t}\right)\|\theta_0\|_2^2,\end{aligned} \tag{5.13}$$

for all $t \geq 0$, by using (5.10).

Then, integrating (5.11) over $[t, t+1]$ shows

$$\int_t^{t+1} \left(\frac{2}{Re} \|\nabla_h \omega\|_2^2 + \epsilon^2 \|\mathbf{u}_z\|_2^2 \right) ds \le \|\omega(t)\|_2^2 + \frac{1}{\epsilon^2} \int_t^{t+1} \|\nabla_h w\|_2^2 ds$$
$$\le C e^{-\frac{1}{\kappa\gamma Re} t} \left(\|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2 \right) + C \left(e^{-\frac{2}{\gamma Pe} t} + e^{-\frac{1}{\kappa\gamma Re} t} \right) \|\theta_0\|_2^2, \quad \text{for all } t \ge 1, \tag{5.14}$$

where the last inequality is due to (5.10) and (5.13).

Furthermore, taking the inner product of (1.5) with $-\Delta_h w$ and using the calculations in (3.20)-(3.22) yield

$$\frac{d}{dt} \|\nabla_h w\|_2^2 + \frac{1}{Re} \|\Delta_h w\|_2^2 \le C \|\omega\|_2^2 (\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2) \|\nabla_h w\|_2^2 + C \left(\|\phi_z\|_2^2 + \|\theta\|_2^2 \right), \tag{5.15}$$

and

$$\|\nabla_h w(t)\|_2^2 + \frac{1}{Re} \int_0^t \|\Delta_h w\|_2^2 ds \le C(\|\nabla_h w_0\|_2, \|\omega_0\|_2, \|\theta_0\|_2), \quad \text{for all } t \ge 0. \tag{5.16}$$

Applying the uniform Gronwall Lemma (see Lemma 2.4) to (5.15), we obtain

$$\|\nabla_h w(t+1)\|_2^2 \le e^{\int_t^{t+1} C \|\omega\|_2^2 (\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2) ds} \left(\int_t^{t+1} \|\nabla_h w\|_2^2 ds + C \int_t^{t+1} \left(\|\phi_z\|_2^2 + \|\theta\|_2^2 \right) ds \right)$$
$$\le \left(e^{-\frac{2}{\gamma Pe} t} + e^{-\frac{1}{\kappa\gamma Re} t} \right) C(\|\mathbf{u}_0\|_2, \|w_0\|_2, \|\theta_0\|_2), \quad \text{for all } t \ge 1, \tag{5.17}$$

where we use (5.3), (5.9)-(5.10) and (5.13)-(5.14). Then, we integrate (5.15) over $[t, t+1]$ to get

$$\frac{1}{Re} \int_t^{t+1} \|\Delta_h w\|_2^2 ds \le \|\nabla_h w(t)\|_2^2 + C \int_t^{t+1} \left[\|\omega\|_2^2 (\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2) \|\nabla_h w\|_2^2 + \|\phi_z\|_2^2 + \|\theta\|_2^2 \right] ds$$
$$\le \left(e^{-\frac{2}{\gamma Pe} t} + e^{-\frac{1}{\kappa\gamma Re} t} \right) C(\|\mathbf{u}_0\|_2, \|w_0\|_2, \|\theta_0\|_2), \quad \text{for all } t \ge 2, \tag{5.18}$$

due to (5.3) and (5.9)-(5.10), (5.13)-(5.14) and (5.17).

Next, we multiply (1.7) by 2θ and integrate the result over $[0, L]^2 \times [0, t]$, then we obtain

$$\frac{1}{2} \overline{\theta^2}(z, t) + \int_0^t \left[\frac{1}{Pe} \overline{|\nabla_h \theta|^2}(z) + \left| \overline{w\theta} \right|^2 (z) \right] ds = \frac{1}{2} \overline{\theta_0^2}(z) \le C \left(\|\theta_0\|_2^2 + \|\partial_z \theta_0\|_2^2 \right), \tag{5.19}$$

for all $t \ge 0$, and for a.e. $z \in [0, 1]$, where we use Lemma 2.2 to obtain the last inequality.

Also, taking the inner product of (1.7) with $-\Delta_h \theta$ and using the same calculation as in (3.25)-(3.27), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla_h \theta\|_2^2 + \frac{1}{Pe}\|\Delta_h \theta\|_2^2 \\
&\le C\|\nabla_h \mathbf{u}\|_2^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2\right)\|\nabla_h \theta\|_2^2 + \|\overline{w\theta}\|_2^2 \|\overline{\theta^2}\|_{L^\infty} + \|\Delta_h w\|_2^2,
\end{aligned}
\tag{5.20}
$$

and

$$
\|\nabla_h \theta(t)\|_2^2 + \frac{1}{Pe}\int_0^t \|\Delta_h \theta\|_2^2 ds \le C(\|\theta_0\|_{H^1}, \|\nabla_h w_0\|_2, \|\omega_0\|_2), \quad \text{for all } t \ge 0. \tag{5.21}
$$

Then, applying the uniform Gronwall Lemma on (5.20), we deduce

$$
\begin{aligned}
\|\nabla_h \theta(t+1)\|_2^2 &\le e^{\int_t^{t+1} C\|\omega\|_2^2(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2)ds} \int_t^{t+1} \left(\|\nabla_h \theta\|_2^2 + \|\Delta_h w\|_2^2 + \|\overline{w\theta}\|_2^2 \|\overline{\theta^2}\|_{L^\infty}\right) ds \\
&\le \left(e^{-\frac{2}{\gamma Pe}t} + e^{-\frac{1}{\kappa\gamma Re}t}\right) C(\|\mathbf{u}_0\|_2, \|w_0\|_2, \|\theta_0\|_2, \|\partial_z \theta_0\|_2), \quad \text{for all } t \ge 2,
\end{aligned}
$$

where we have used (5.4), (5.9), (5.13)-(5.14), (5.18) and (5.19).

It remains to show that $\|\mathbf{u}_z\|_2^2 + \|w_z\|_2^2 + \|\theta_z\|_2^2$ grows at most exponentially in time. Indeed, performing similar calculations as in section 3.2.7, we obtain

$$
\|\partial_z w(t)\|_2^2 + \|\partial_z \mathbf{u}(t)\|_2^2 + \|\partial_z \theta(t)\|_2^2 \le \left(\|\partial_z w_0\|_2^2 + \|\partial_z \mathbf{u}_0\|_2^2 + C\|\partial_z \theta_0\|_2^2\right) e^{\mathcal{M}(t)}, \tag{5.22}
$$

where

$$
\begin{aligned}
\mathcal{M}(t) = C\int_0^t &\Big(\|\Delta_h w\|_2^2 + \|\mathbf{u}_z\|_2^2 + \|\omega\|_2^2\|\mathbf{u}\|_2^2 + \|\omega\|_2^2\|\mathbf{u}_z\|_2^2 \\
&+ \|\Delta_h \theta\|_2^2 + \|\overline{\theta^2}\|_{L^\infty} + \sup_{z\in[0,1]} \int_0^t \left|\overline{w\theta}\right|^2 d\tau + 1\Big) ds \\
&\le C(\|\omega_0\|_2, \|\nabla_h w_0\|_2, \|\theta_0\|_{H^1}) + C\left(\|\theta_0\|_2^2 + \|\partial_z \theta_0\|_2^2 + 1\right) t,
\end{aligned}
\tag{5.23}
$$

for all $t \ge 0$, where we use (5.6), (5.12), (5.16), (5.19) and (5.21) to obtain the last inequality. Notice that (5.22)-(5.23) implies (1.22). The proof for Theorem 1.3 is complete.

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