Z. Angew. Math. Phys. 64 (2013), 621–658

0044-2275/13/030621-38
published online August 10, 2012
DOI 10.1007/s00033-012-0252-6

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global existence and decay of energy to systems of wave equations with damping and supercritical sources

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Abstract. This paper is concerned with a system of nonlinear wave equations with *supercritical interior and boundary sources* and subject to interior and boundary damping terms. It is well-known that the presence of a nonlinear boundary source causes significant difficulties since the linear Neumann problem for the single wave equation is not, in general, well-posed in the finite-energy space $H^1(\Omega) \times L^2(\partial\Omega)$ with boundary data from $L^2(\partial\Omega)$ (due to the failure of the uniform Lopatinskii condition). Additional challenges stem from the fact that the sources considered in this article are *non-dissipative* and are not locally Lipschitz from $H^1(\Omega)$ into $L^2(\Omega)$ or $L^2(\partial\Omega)$. With some restrictions on the parameters in the system and with careful analysis involving the Nehari Manifold, we obtain global existence of a unique weak solution and establish (depending on the behavior of the dissipation in the system) exponential and algebraic *uniform decay rates of energy.* Moreover, we prove a blow-up result for weak solutions with *nonnegative initial energy.*

Mathematics Subject Classification (2010). Primary 35L05 · 35L20; Secondary 58J45.

Keywords. Wave equations · Damping and source terms · Potential well · Nehari manifold · Global existence · Energy decay rates · Blow up.

1. Introduction

1.1. Preliminaries

Over the recent years, wave equations under the influence of nonlinear damping and nonlinear sources have generated considerable interest. Of central interest is the analysis of how two competing forces (nonlinear damping and source terms) influence the behavior of solutions. Many results [1–3, 14, 26, 28–30] have been established when the sources in the system are subcritical or critical. In this case, the sources are locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Omega)$ and into $L^2(\partial\Omega)$, and thus, obtaining existence of local solutions can achieved via Galerkin approximations or standard fixed point theorems. However, very few articles ([8–11] and most recently in [13, 15, 16, 27]) addressed wave equations influenced by supercritical sources.

For the sake of clarity, we restrict our analysis to the physically more relevant case when $\Omega \subset \mathbb{R}^3$. Our results extend easily to bounded domains in $\mathbb{R}^n$, by accounting for the corresponding Sobolev imbeddings, and accordingly adjusting the conditions imposed on the parameters. Thus, throughout the paper, we assume that Ω is bounded, open, and connected non-empty set in $\mathbb{R}^3$ with a smooth boundary $\Gamma = \partial\Omega$. In this paper, we study the following system of wave equations:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f_1(u,v) & \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u,v) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u + u + g(u_t) = h(u) & \text{on } \Gamma \times (0,\infty), \\ v = 0 & \text{on } \Gamma \times (0,\infty), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H^1_0(\Omega), v_t(0) = v_1 \in L^2(\Omega), \end{cases} \tag{1.1}$$

where the nonlinearities $f_1(u,v)$, $f_2(u,v)$, and $h(u)$ are supercritical interior and boundary sources, and the damping functions g_1, g_2, and g are arbitrary continuous monotone increasing graphs vanishing at the origin.

Some special cases of (1.1) arise in quantum field theory. In particular, Segal [33] introduced the system

$$u_{tt} - \Delta u = -\alpha_1^2 u - \beta_1^2 v^2 u, \quad v_{tt} - \Delta v = -\alpha_2^2 v - \beta_2^2 u^2 v,$$

as a model to describe the interaction of scalar fields u, v of masses α_1, α_2, respectively, subject to the interaction constants β_1 and β_2. This system defines the motion of charged mesons in an electromagnetic field. Later, Makhankov [23] pointed out some essential properties of such interacting relativistic fields. On the other hand, coupled wave equations arise naturally in investigating longitudinal dynamical effects in classical semi-conductor lasers and nonlinear optics [4,35,37]. Moreover, nonlinear systems of coupled wave equations have been derived from Maxwell's equations for an electromagnetic field in a periodically modulated waveguide under the assumption that transversal and longitudinal effects can be separated [4]. Thus, the source-damping interaction in (1.1) encompasses a broad class of problems in quantum field theory and certain mechanical applications [17,24,34]. For instance, a relevant model to (1.1) is the Reissner–Mindlin plate equations (see for instance, Ch. 3 in [18]), which consist of three coupled PDE's: a wave equations and two wave-like equations, where each equation is influenced by nonlinear damping and source terms. It is worth noting that *non-dissipative* "energy-building" sources, especially those on the boundary, arise when one considers a wave equation being coupled with other types of dynamics, such as structure-acoustic or fluid-structure interaction models (Lasiecka [20]). In light of these applications, we are mainly interested in higher-order nonlinearities, as described in following assumption.

Assumption 1.1.

- ***Interior sources*** **:** $f_j(u,v) \in C^1(\mathbb{R}^2)$ *such that*
$$|\nabla f_j(u,v)| \le C(|u|^{p-1} + |v|^{p-1} + 1), \ \ j = 1,2, \ \ \textit{where } \ 1 \le p < 6.$$
- ***Boundary source*** **:** $h \in C^1(\mathbb{R})$ *such that*
$$|h'(s)| \le C(|s|^{k-1} + 1), \ \ \textit{where } \ 1 \le k < 4.$$
- ***Damping*** **:** g_1, g_2, *and* g *are continuous and monotone increasing functions on* $\mathbb{R}$ *with* $g_1(0) = g_2(0) = g(0) = 0$. *In addition, the following growth conditions hold: there exist positive constants* a_j *and* b_j, $j = 1,2,3$, *such that, for* $|s| \ge 1$,
$$a_1|s|^{m+1} \le g_1(s)s \le b_1|s|^{m+1}, \ \ \textit{where } \ m \ge 1,$$
$$a_2|s|^{r+1} \le g_2(s)s \le b_2|s|^{r+1}, \ \ \textit{where } \ r \ge 1,$$
$$a_3|s|^{q+1} \le g(s)s \le b_3|s|^{q+1}, \ \ \textit{where } \ q \ge 1.$$
- ***Parameters*** **:** $\max\{p\frac{m+1}{m}, p\frac{r+1}{r}\} < 6$; $k\frac{q+1}{q} < 4$.

We note here that in Assumption 1.1 and throughout the paper, all generic constants will be denoted by C, and they may change from line-to-line.

1.2. Literature overview

Wellposedness and asymptotic behavior of wave equations with at most *critical* semi-linear nonlinearities have been extensively studied, and by now, the established results forms a comprehensive theory. More recent research efforts aim at the more challenging class of models with higher-order nonlinearities, such as supercritical and super-supercritical sources.

In the presence of such strong nonlinearities, the local solvability becomes much harder to establish. For a single wave equation, substantial advancements have been made by Bociu and Lasiecka in a series of papers [8–11]. Indeed, the recent results by Bociu and Lasiecka included local and global existence, uniqueness, continuous dependence on initial data, and some blow-up results for wave equations on bounded domains subject to super-supercritical sources and damping terms (acting both on the boundary and in the interior of the domain). These techniques have been also used to establish similar results for the Cauchy problem of a single wave equation [12]. Subsequently, relying on this well-posedness theory, the authors of [7] have investigated the long-term behavior and uniform decay rates for solutions confined to a potential well. For other related results on potential well solutions, see [2,22,25,38,39] and the references therein.

A well-known system, which is a special case of (1.1), is the following polynomially damped system that has been studied extensively in the literature [1,2,28,29]:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v) & \text{in } \Omega \times (0,T), \\ u = v = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{1.2}$$

where the sources f_1, f_2 are very specific functions. Namely, $f_1(u,v) = \partial_u F(u,v)$ and $f_2(u,v) = \partial_v F(u,v)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a homogeneous C^1-function given by:

$$F(u,v) = a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}, \tag{1.3}$$

where $p \geq 3$, $a > 1$, and $b > 0$.

Systems of nonlinear wave equations such as (1.2) go back to Reed [31] who proposed a similar system in three space dimensions but without the presence of damping. Indeed, recently in [1] and later in [2], the authors studied system (1.2) with Dirichlét boundary conditions on both u and v where the exponent of the source was restricted to be *critical* ($p = 3$ in 3D). More recently, the authors of [15,16] (following the strategy developed in [8–11]) studied the more general system (1.1) and obtained several results on the existence of local and global weak solutions, uniqueness, continuous dependence on initial data, and blow up in finite time for the larger range of the exponent p: supercritical sources ($3 < p \leq 5$) and super-supercritical ($5 < p < 6$). The main tools for proving local existence in [16] were nonlinear semi-groups and monotone operator theory. Another crucial ingredient to the local solvability in [16] is the recent results in [5], where the authors of [5] resolved the question of the identification of the subdifferential of a sum of two convex functionals (one is originating from the interior and the other from the boundary damping) without imposing any growth restrictions on the defining convex functions.

1.3. New goals and challenges

The main goal of the present paper is to complement the results of [15,16] by establishing global existence of potential well solutions, uniform decay rates of energy, and blow up of solutions with non-negative initial energy. Comparing with the results of [2] for system (1.2) with $p = 3$, our results extend and refine the results of [2] in the following sense: (i) System (1.1) is more general than (1.2) with supercritical sources and subject to a nonlinear Robin boundary condition. However, we note here that the mixture of Robin and Dirichlét boundary conditions in system (1.1) is not essential to the methods used in this paper nor to our results. Indeed, all of our results in this paper can be easily obtained if instead one imposes Robin boundary conditions on both u and v. (ii) The global existence and energy decay results in [2] are obtained only when the exponents of the damping functions are restricted to the case m, $r \leq 5$. Here, we allow m, r to be larger than 5, provided we impose additional assumptions on the regularity of weak solutions. (iii) In addition to the standard case $p > \max\{m,r\}$ and $k > q$ for our blow-up result, we consider another scenario in which the interior source is more dominant than both feedback mappings in the interior and on the boundary. Specifically, we prove a blow-up result in the case $p > \max\{m,r,2q-1\}$, and without the additional assumption $k > q$. Although this kind of blow-up result has been established for solutions with *negative* initial energy [9,15], to our knowledge, our result is new for wave equations with *non-negative* initial energy.

Our strategy for the blow-up results in this paper follows the general framework of [2,7]. However, our proofs had to be significantly adjusted to accommodate the coupling in the system (1.1) and the new case $p > \max\{m, r, 2q-1\}$. For the decay of energy, we follow the roadmap paper by Lasiecka and Tataru [19] and its refined versions in [2,7,21,36], which involve comparing the energy of the system to a suitable ordinary differential equation. It is worth mentioning that the effect of quasilinear damping terms in (1.1) leads to highly non-trivial long-time behavior of solutions. It is known that super-linear stabilizing feedbacks may slow down the energy decay to algebraic or logarithmic rates [21]. On the another hand, there are no known uniform decay results for some problems with degenerate damping, such as the one in [6].

1.4. Outline

The paper is organized as follows. In Sect. 2, we begin by citing the local-wellposedness results established in [16]. Subsequently, we revisit the potential well theory and the strong connection of (1.1) with the elliptic theory. The statements of the main results: global existence of potential well solutions, uniform energy decay rates, and blow up of solutions with non-negative initial energy are summarized in Sect. 2. Global existence is then proved in Sect. 3. In Sect. 4, we prove the uniform energy decay rates of energy, where the analysis is divided into several parts. Finally, Sect. 5 is devoted to the proof of our blow-up result.

2. Preliminaries and main results

We begin by introducing the following notations that will be used throughout the paper:

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \ |u|_s = \|u\|_{L^s(\Gamma)}, \ \|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)};$$
$$(u,v)_\Omega = (u,v)_{L^2(\Omega)}, \ (u,v)_\Gamma = (u,v)_{L^2(\Gamma)}, \ (u,v)_{1,\Omega} = (u,v)_{H^1(\Omega)}.$$

We also use the notation γu to denote the *trace* of u on Γ, and we write $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma u(t))$ as γu_t. In addition, we note that $(\|\nabla u\|_2^2 + |\gamma u|_2^2)^{1/2}$ is equivalent to the standard $H^1(\Omega)$ norm. This fact follows from a Poincaré–Wirtinger type of inequality:

$$\|u\|_2^2 \le c_0 \left(\|\nabla u\|_2^2 + |\gamma u|_2^2\right), \ \text{ for all } u \in H^1(\Omega). \tag{2.1}$$

Thus, throughout the paper, we put

$$\|u\|_{1,\Omega}^2 = \|\nabla u\|_2^2 + |\gamma u|_2^2 \ \text{ and } \ (u,v)_{1,\Omega} = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma,$$

for $u, v \in H^1(\Omega)$.

For the reader's convenience, we begin by citing some of the main results in [16] that are essential to the results of this paper. To do so, we first introduce the definition of a weak solution.

Definition 2.1. *A pair of functions* (u, v) *is said to be a* weak solution *of (1.1) on* $[0, T]$ *if*

- $u \in C([0,T]; H^1(\Omega))$, $v \in C([0,T]; H_0^1(\Omega))$, $u_t \in C([0,T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$, $\gamma u_t \in L^{q+1}(\Gamma \times (0,T))$, $v_t \in C([0,T]; L^2(\Omega)) \cap L^{r+1}(\Omega \times (0,T))$;
- $(u(0), v(0)) = (u_0, v_0) \in H^1(\Omega) \times H_0^1(\Omega)$, $(u_t(0), v_t(0)) = (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$;

- *For all $t \in [0,T]$, u and v verify the following identities:*

$$(u_t(t),\phi(t))_\Omega - (u_t(0),\phi(0))_\Omega + \int_0^t [-(u_t(\tau),\phi_t(\tau))_\Omega + (u(\tau),\phi(\tau))_{1,\Omega}]\,d\tau$$
$$+ \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau)\,dx\,d\tau + \int_0^t \int_\Gamma g(\gamma u_t(\tau))\gamma\phi(\tau)\,d\Gamma\,d\tau$$
$$= \int_0^t \int_\Omega f_1(u(\tau),v(\tau))\phi(\tau)\,dx\,d\tau + \int_0^t \int_\Gamma h(\gamma u(\tau))\gamma\phi(\tau)\,d\Gamma\,d\tau, \tag{2.2}$$

$$(v_t(t),\psi(t))_\Omega - (v_t(0),\psi(0))_\Omega + \int_0^t [-(v_t(\tau),\psi_t(\tau))_\Omega + (v(\tau),\psi(\tau))_{1,\Omega}]\,d\tau$$
$$+ \int_0^t \int_\Omega g_2(v_t(\tau))\psi(\tau)\,dx\,d\tau = \int_0^t \int_\Omega f_2(u(\tau),v(\tau))\psi(\tau)\,dx\,d\tau, \tag{2.3}$$

for all test functions satisfying: $\phi \in C([0,T];H^1(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$ such that $\gamma\phi \in L^{q+1}(\Gamma \times (0,T))$ with $\phi_t \in L^1([0,T];L^2(\Omega))$ and $\psi \in C([0,T];H^1_0(\Omega)) \cap L^{r+1}(\Omega \times (0,T))$ such that $\psi_t \in L^1([0,T];L^2(\Omega))$.

As mentioned earlier, our work in this paper is based on the existence results, which was established in [16].

Theorem 2.2. (Local and global weak solutions [16]) *Assume the validity of the Assumption 1.1. Then, there exists a local weak solution (u,v) to (1.1) defined on $[0,T]$, for some $T > 0$. Moreover, we have:*

- *(u,v) satisfies the following energy identity for all $t \in [0,T]$:*

$$\mathscr{E}(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma\mathrm{d}\tau$$
$$= \mathscr{E}(0) + \int_0^t \int_\Omega [f_1(u,v)u_t + f_2(u,v)v_t]\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma u_t \mathrm{d}\Gamma\mathrm{d}\tau, \tag{2.4}$$

where the quadratic energy is given by

$$\mathscr{E}(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right). \tag{2.5}$$

- *If, in addition, we assume $p \leq \min\{m,r\}$, $k \leq q$ and $u_0, v_0 \in L^{p+1}(\Omega)$, $\gamma u_0 \in L^{k+1}(\Gamma)$, then the said solution (u,v) is a global weak solution and T can be taken arbitrarily large.*

Remark 2.3. *Under additional assumptions on the sources and the boundary damping, uniqueness of weak solutions for (1.1) has been established in [16]. Moreover, the results of [15] show that every weak solution of (1.1) with* negative *initial energy blows up in finite time, provided either: $p > \max\{m,r\}$ and $k > q$, or $p > \max\{m,r,2q-1\}$. We refer the reader to [15,16] for complete statements of these results.*

2.1. Potential well

In this section, we begin by briefly pointing out the connection of problem (1.1) to some important aspects of the theory of elliptic equations. In order to do so, we need to impose additional assumptions on the interior sources f_1, f_2, and boundary source h.

Assumption 2.4.

- *There exists a nonnegative function* $F(u,v) \in C^1(\mathbb{R}^2)$ *such that* $\partial_u F(u,v) = f_1(u,v)$, $\partial_v F(u,v) = f_2(u,v)$, *and* F *is homogeneous of order* $p+1$, *that is,* $F(\lambda u, \lambda v) = \lambda^{p+1} F(u,v)$, *for all* $\lambda > 0$, $(u,v) \in \mathbb{R}^2$.
- *There exists a nonnegative function* $H(s) \in C^1(\mathbb{R})$ *such that* $H'(s) = h(s)$, *and* H *is homogeneous of order* $k+1$, *that is,* $H(\lambda s) = \lambda^{k+1} H(s)$, *for all* $\lambda > 0$, $s \in \mathbb{R}$.

Remark 2.5. *We note that the special function* $F(u,v)$ *defined in (1.3) satisfies Assumption 2.4, provided* $p \geq 3$. *However, there is a large class of functions that satisfies Assumption 2.4. For instance, functions of the form (with an appropriate range of values for* p, s, *and* σ*):*

$$\mathcal{F}(u,v) = a|u|^{p+1} + b|v|^{p+1} + \alpha |u|^s |v|^{p+1-s} + \beta (|u|^\sigma + |v|^\sigma)^{\frac{p+1}{\sigma}},$$

satisfy Assumption 2.4. Moreover, since F *and* H *are homogeneous, then the Euler homogeneous function theorem gives the following useful identities:*

$$f_1(u,v)u + f_2(u,v)v = (p+1)F(u,v) \;\; \text{and} \;\; h(s)s = (k+1)H(s). \tag{2.6}$$

Finally, we note that the assumptions $|\nabla f_j(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, $j = 1,2$, *and* $|h'(s)| \leq C(|s|^{k-1} + 1)$ *(as required by Assumption 1.1) imply that there exists a constant* $M > 0$ *such that* $F(u,v) \leq M(|u|^{p+1} + |v|^{p+1} + 1)$ *and* $H(s) \leq M(|s|^{k+1} + 1)$, *for all* $u, v, s \in \mathbb{R}$. *Therefore, by the homogeneity of* F *and* H, *we must have*

$$F(u,v) \leq M(|u|^{p+1} + |v|^{p+1}) \;\; \text{and} \;\; H(s) \leq M|s|^{k+1}. \tag{2.7}$$

We start by defining the *total energy* of the system (1.1) as follows:

$$\begin{aligned} E(t) :=& \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) \\ & - \int_\Omega F(u(t), v(t)) \mathrm{d}x - \int_\Gamma H(\gamma u(t)) \mathrm{d}\Gamma. \end{aligned} \tag{2.8}$$

Put $X := H^1(\Omega) \times H_0^1(\Omega)$ and define the functional $J : X \to \mathbb{R}$ by:

$$J(u,v) := \frac{1}{2}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - \int_\Omega F(u,v)\mathrm{d}x - \int_\Gamma H(\gamma u)\mathrm{d}\Gamma, \tag{2.9}$$

where $J(u,v)$ represents the *potential energy* of the system. Therefore, the total energy can be written as:

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) + J(u(t), v(t)). \tag{2.10}$$

In addition, simple calculations shows that the Fréchet derivative of J at $(u,v) \in X$ is given by:

$$\begin{aligned} \langle J'(u,v), (\phi, \psi) \rangle =& \int_\Omega \nabla u \cdot \nabla \phi \mathrm{d}x + \int_\Gamma \gamma u \gamma \phi \mathrm{d}\Gamma + \int_\Omega \nabla v \cdot \nabla \psi \mathrm{d}x \\ & - \int_\Omega [f_1(u,v)\phi + f_2(u,v)\psi]\mathrm{d}x - \int_\Gamma h(\gamma u)\gamma\phi \mathrm{d}\Gamma, \end{aligned} \tag{2.11}$$

for all $(\phi, \psi) \in X$.

Associated to the functional J is the well-known *Nehari manifold*, namely

$$\mathcal{N} := \{(u,v) \in X\backslash\{(0,0)\} : \langle J'(u,v),(u,v)\rangle = 0\}. \tag{2.12}$$

It follows from (2.11) and (2.6) that the Nehari manifold can be put as:

$$\begin{aligned}\mathcal{N} = \Big\{&(u,v) \in X\backslash\{(0,0)\} : \\ &\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 = (p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma\Big\}.\end{aligned} \tag{2.13}$$

In order to introduce the potential well, we first prove the following lemma.

Lemma 2.6. *In addition to Assumptions 1.1 and 2.4, further assume that* $1 < p \leq 5$ *and* $1 < k \leq 3$. *Then,*

$$d := \inf_{(u,v)\in\mathcal{N}} J(u,v) > 0. \tag{2.14}$$

Proof. Fix $(u,v) \in \mathcal{N}$. Then, it follows from (2.9) and (2.13) that

$$J(u,v) \geq \left(\frac{1}{2} - \frac{1}{c}\right)\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right). \tag{2.15}$$

where $c := \min\{p+1, k+1\} > 2$. Since $(u,v) \in \mathcal{N}$, then the bounds (2.7) yield

$$\begin{aligned}\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 &\leq C_{p,k}\left(\int_\Omega (|u|^{p+1} + |v|^{p+1})\mathrm{d}x + \int_\Gamma |\gamma u|^{k+1}\mathrm{d}\Gamma\right) \\ &\leq C\left(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1} + \|u\|_{1,\Omega}^{k+1}\right).\end{aligned} \tag{2.16}$$

Thus,

$$\|(u,v)\|_X^2 \leq C\left(\|(u,v)\|_X^{p+1} + \|(u,v)\|_X^{k+1}\right),$$

and since $(u,v) \neq (0,0)$, we have

$$1 \leq C\left(\|(u,v)\|_X^{p-1} + \|(u,v)\|_X^{k-1}\right).$$

It follows that $\|(u,v)\|_X \geq s_1 > 0$ where s_1 is the unique positive solution of the equation $C(s^{p-1} + s^{k-1}) = 1$, where p, $k > 1$. Then, by (2.15), we arrive at

$$J(u,v) \geq \left(\frac{1}{2} - \frac{1}{c}\right)s_1^2$$

for all $(u,v) \in \mathcal{N}$. Thus, (2.14) follows. □

As in [2], we introduce the following sets:

$$\begin{aligned}\mathcal{W} &:= \{(u,v) \in X : J(u,v) < d\}, \\ \mathcal{W}_1 &:= \left\{(u,v) \in \mathcal{W} : \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 > (p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma\right\} \\ &\qquad \cup \{(0,0)\}, \\ \mathcal{W}_2 &:= \left\{(u,v) \in \mathcal{W} : \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 < (p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma\right\}.\end{aligned}$$

Clearly, $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}$. In addition, we refer to $\mathcal{W}$ as the *potential well* and d as the *depth* of the well. The set $\mathcal{W}_1$ is regarded as the "good" part of the well, as we will show that every weak solution exists globally in time, provided the initial data are taken from $\mathcal{W}_1$ and the initial energy is under the level d. On the other hand, if the initial data are taken from $\mathcal{W}_2$ and the sources dominate the damping, we will prove a blow-up result for weak solutions with nonnegative initial energy.

The following lemma will be needed in the sequel.

Lemma 2.7. *Under the assumptions of Lemma 2.6, the depth of the potential well d coincides with the mountain pass level. Specifically,*

$$d = \inf_{(u,v)\in X\setminus\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)). \tag{2.17}$$

Proof. Recall $X = H^1(\Omega)\times H_0^1(\Omega)$. Let $(u,v)\in X\setminus\{(0,0)\}$ be fixed. By recalling Assumption 2.4, it follows that,

$$J(\lambda(u,v)) = \frac{1}{2}\lambda^2\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - \lambda^{p+1}\int\limits_\Omega F(u,v)\mathrm{d}x - \lambda^{k+1}\int\limits_\Gamma H(\gamma u)\mathrm{d}\Gamma, \tag{2.18}$$

for $\lambda\geq 0$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda(u,v)) =& \Big[\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - (p+1)\lambda^{p-1}\int\limits_\Omega F(u,v)\mathrm{d}x \\ &- (k+1)\lambda^{k-1}\int\limits_\Gamma H(\gamma u)\mathrm{d}\Gamma\Big].\end{aligned} \tag{2.19}$$

Hence, the only critical point in $(0,\infty)$ for the mapping $\lambda\mapsto J(\lambda(u,v))$ is λ_0 that satisfies the equation:

$$\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) = (p+1)\lambda_0^{p-1}\int\limits_\Omega F(u,v)\mathrm{d}x + (k+1)\lambda_0^{k-1}\int\limits_\Gamma H(\gamma u)\mathrm{d}\Gamma. \tag{2.20}$$

Moreover, it is easy to see that

$$\sup_{\lambda\geq 0} J(\lambda(u,v)) = J(\lambda_0(u,v)). \tag{2.21}$$

By the definition of $\mathcal{N}$ and noting (2.20), we conclude that $\lambda_0(u,v)\in\mathcal{N}$. As a result,

$$J(\lambda_0(u,v)) \geq \inf_{(y,z)\in\mathcal{N}} J(y,z) = d. \tag{2.22}$$

By combining (2.21) and (2.22), one has

$$\inf_{(u,v)\in X\setminus\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)) \geq d. \tag{2.23}$$

On the other hand, for each fixed $(y,z)\in\mathcal{N}$, we find that (using (2.13) and (2.20)) the only critical point in $(0,\infty)$ of the mapping $\lambda\mapsto J(\lambda(y,z))$ is $\lambda_0 = 1$. Therefore, $\sup_{\lambda\geq 0} J(\lambda(y,z)) = J(y,z)$ for each $(y,z)\in\mathcal{N}$. Hence,

$$\inf_{(u,v)\in X\setminus\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)) \leq \inf_{(y,z)\in\mathcal{N}} \sup_{\lambda\geq 0} J(\lambda(y,z)) = \inf_{(y,z)\in\mathcal{N}} J(y,z) = d. \tag{2.24}$$

Combining (2.23) and (2.24) gives the desired result (2.17). ∎

2.2. Main results

Our first result establishes the existence of a global weak solution to (1.1), provided the initial data come from $\mathcal{W}_1$ and the initial energy is less than d, and without imposing the conditions $p \leq \min\{m, r\}$, $k \leq q$, as required by Theorem 2.2.

In order to state our first result, we recall the quadratic energy $\mathscr{E}(t)$ and the total energy $E(t)$ as defined in (2.5) and (2.8), respectively.

Theorem 2.8. (Global Solutions) *In addition to Assumptions 1.1 and 2.4, further assume* $(u_0, v_0) \in \mathcal{W}_1$ *and* $E(0) < d$. *If* $1 < p \leq 5$ *and* $1 < k \leq 3$, *then the weak solution* (u, v) *of (1.1) is a global solution. Furthermore, we have:*

$$(u(t), v(t)) \in \mathcal{W}_1,$$

$$\mathscr{E}(t) < d\left(\frac{c}{c-2}\right), \tag{2.25}$$

$$\left(1 - \frac{2}{c}\right)\mathscr{E}(t) \leq E(t) \leq \mathscr{E}(t), \tag{2.26}$$

for all $t \geq 0$, *where* $c = \min\{p+1, k+1\} > 2$.

Since the weak solution furnished by Theorem 2.8 is a global solution and the total energy $E(t)$ remains positive for all $t \geq 0$, we may study the uniform decay rates of the energy. Specifically, we will show that if the initial data come from a **closed** subset of $\mathcal{W}_1$, then the energy $E(t)$ decays either exponentially or algebraically, depending on the behaviors of the functions g_1, g_2, and g near the origin.

In order to state our result on the energy decay, we need some preparations. Define the function

$$\mathcal{G}(s) := \frac{1}{2}s^2 - MR_1 s^{p+1} - MR_2 s^{k+1}, \tag{2.27}$$

where the constant $M > 0$ is as given in (2.7) and

$$R_1 := \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|u\|_{p+1}^{p+1}}{\|u\|_{1,\Omega}^{p+1}}, \quad R_2 := \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{|\gamma u|_{k+1}^{k+1}}{\|u\|_{1,\Omega}^{k+1}}. \tag{2.28}$$

Since $p \leq 5$ and $k \leq 3$, by Sobolev Imbedding Theorem, we know $0 < R_1$, $R_2 < \infty$.

A straightforward calculation shows that $\mathcal{G}'(s)$ has a unique positive zero, say at $s_0 > 0$, and

$$\sup_{s \in [0,\infty)} \mathcal{G}(s) = \mathcal{G}(s_0).$$

Thus, we define the set

$$\tilde{\mathcal{W}}_1 := \{(u, v) \in X : \|(u, v)\|_X < s_0,\ J(u, v) < \mathcal{G}(s_0)\}. \tag{2.29}$$

We will show in Proposition 4.2 that $\mathcal{G}(s_0) \leq d$ and $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$.

Furthermore, for each fixed small value $\delta > 0$, we define a closed subset of $\tilde{\mathcal{W}}_1$, namely

$$\tilde{\mathcal{W}}_1^\delta := \{(u, v) \in X : \|(u, v)\|_X \leq s_0 - \delta,\ J(u, v) \leq \mathcal{G}(s_0 - \delta)\}. \tag{2.30}$$

Indeed, we will show in Proposition 4.3 that $\tilde{\mathcal{W}}_1^\delta$ is invariant under the dynamics, if the initial energy satisfies $E(0) \leq \mathcal{G}(s_0 - \delta)$.

The following theorem addresses the uniform decay rates of energy. In the standard case m, $r \leq 5$, and $q \leq 3$, we do not impose any additional assumptions on the weak solutions furnished by Theorem 2.8. However, if any of the exponents of damping is *large*, then we need additional assumptions on the regularity of weak solutions. More precisely, we have the following result.

Theorem 2.9. (Uniform Decay Rates) *In addition to Assumptions 1.1 and 2.4, further assume:* $1 < p < 5$, $1 < k < 3$, $u_0 \in L^{m+1}(\Omega)$, $v_0 \in L^{r+1}(\Omega)$, $\gamma u_0 \in L^{q+1}(\Gamma)$, $(u_0, v_0) \in \tilde{\mathcal{W}}_1^{\delta}$, *and* $E(0) < \mathcal{G}(s_0 - \delta)$ *for some* $\delta > 0$. *In addition, assume* $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ *if* $m > 5$, $v \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(r-1)}(\Omega))$ *if* $r > 5$, *and* $\gamma u \in L^\infty(\mathbb{R}^+; L^{2(q-1)}(\Gamma))$ *if* $q > 3$, *where* (u, v) *is the global solution of (1.1) furnished by Theorem 2.8.*

- *If* g_1, g_2, *and* g *are linearly bounded near the origin, then the total energy* $E(t)$ *decays exponentially:*

$$E(t) \leq \tilde{C}E(0)\mathrm{e}^{-wt}, \quad \text{for all } t \geq 0, \tag{2.31}$$

  where $\tilde{C}$ *and* w *are positive constants.*
- *If at least one of the feedback mappings* g_1, g_2, *and* g *is not linearly bounded near the origin, then* $E(t)$ *decays algebraically:*

$$E(t) \leq C(E(0))(1+t)^{-\beta}, \quad \text{for all } t \geq 0, \tag{2.32}$$

  where $\beta > 0$ *(specified in (4.11)) depends on the growth rates of* g_1, g_2 *and* g *near the origin.*

Our final result addresses the blow up of potential well solutions with *non-negative* initial energy. It is important to note that the blow-up result in [15] deals with the case of *negative* initial energy for general weak solutions (not necessarily potential well solutions).

Theorem 2.10. (Blow up of Solutions) *In addition to Assumptions 1.1 and 2.4, further assume for* <u>*all*</u> $s \in \mathbb{R}$,

$$\begin{aligned} &a_1|s|^{m+1} \leq g_1(s)s \leq b_1|s|^{m+1}, \ \text{ where } m \geq 1, \\ &a_2|s|^{r+1} \leq g_2(s)s \leq b_2|s|^{r+1}, \ \text{ where } r \geq 1, \\ &a_3|s|^{q+1} \leq g(s)s \leq b_3|s|^{q+1}, \ \text{ where } q \geq 1. \end{aligned} \tag{2.33}$$

In addition, we suppose $F(u, v) \geq \alpha_0(|u|^{p+1} + |v|^{p+1})$, *for some* $\alpha_0 > 0$, *and* $H(s) > 0$, *for all* $s \neq 0$. *If* $1 < p \leq 5$, $1 < k \leq 3$, $(u_0, v_0) \in \mathcal{W}_2$, $0 \leq E(0) < \rho d$, *where*

$$\rho := \frac{\min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\}}{\max\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\}} \leq 1, \tag{2.34}$$

then, the weak solution (u, v) *of (1.1) (as furnished by Theorem 2.2) blows up in finite time, provided either*

- $p > \max\{m, r\}$ *and* $k > q$,
  or
- $p > \max\{m, r, 2q - 1\}$.

Remark 2.11. *The blow-up result in Theorem 2.10 relies on the blow-up result in [15] for negative initial energy. Therefore, as in [15], we conclude from Theorem 2.10 that*

$$\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty,$$

as $t \to T^-$, *for some* $0 < T < \infty$.

3. Global solutions

This section is devoted to the proof of Theorem 2.8.

Proof. The argument will be carried out in two steps.

Step 1. We first show the invariance of $\mathcal{W}_1$ under the dynamics, that is, $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$, where $[0, T)$ is the maximal interval of existence.

Notice the energy identity (2.4) is equivalent to

$$E(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma\mathrm{d}\tau = E(0). \tag{3.1}$$

Since g_1, g_2, and g are all monotone increasing, then it follows from the regularity of the solutions (u, v) that

$$E'(t) = -\int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\mathrm{d}x - \int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma \leq 0. \tag{3.2}$$

Thus,

$$J(u(t), v(t)) \leq E(t) \leq E(0) < d, \qquad \text{for all } t \in [0, T). \tag{3.3}$$

It follows that $(u(t), v(t)) \in \mathcal{W}$ for all $t \in [0, T)$.

To show that $(u(t), v(t)) \in \mathcal{W}_1$ on $[0, T)$, we proceed by contradiction. Assume that there exists $t_1 \in (0, T)$ such that $(u(t_1), v(t_1)) \notin \mathcal{W}_1$. Since $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, then it must be the case that $(u(t_1), v(t_1)) \in \mathcal{W}_2$.

Let us show now that the function $t \mapsto \int_\Omega F(u(t), v(t))\mathrm{d}x$ is continuous on $[0, T)$. Indeed, since $|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, it follows that $|f_j(u, v)| \leq C(|u|^p + |v|^p + 1)$, $j = 1, 2$. By recalling that F is homogeneous of order $p+1$, one has $f_j(u, v)$ are homogeneous of order p, $j = 1, 2$. Therefore,

$$|f_j(u, v)| \leq C(|u|^p + |v|^p), \quad j = 1, 2. \tag{3.4}$$

Fix an arbitrary $t_0 \in [0, T)$. By the Mean Value Theorem and (3.4), we have

$$\begin{aligned} &\int_\Omega |F(u(t), v(t)) - F(u(t_0), v(t_0))|\mathrm{d}x \\ &\quad \leq C \int_\Omega \left(|u(t)|^p + |v(t)|^p + |u(t_0)|^p + |v(t_0)|^p\right)\left(|u(t) - u(t_0)| + |v(t) - v(t_0)|\right)\mathrm{d}x \\ &\quad \leq C \left(\|u(t)\|^p_{\frac{6}{5}p} + \|v(t)\|^p_{\frac{6}{5}p} + \|u(t_0)\|^p_{\frac{6}{5}p} + \|v(t_0)\|^p_{\frac{6}{5}p}\right)\left(\|u(t) - u(t_0)\|_6 + \|v(t) - v(t_0)\|_6\right). \end{aligned} \tag{3.5}$$

Since $p \leq 5$, we know $\frac{6}{5}p \leq 6$, so by the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the regularity of the weak solution $(u, v) \in C([0, T); H^1(\Omega) \times H^1_0(\Omega))$, we obtain from (3.5) that

$$\lim_{t \to t_0} \int_\Omega |F(u(t), v(t)) - F(u(t_0), v(t_0))|\mathrm{d}x = 0,$$

that is, $\int_\Omega F(u(t), v(t))\mathrm{d}x$ is continuous on $[0, T)$.

Likewise, the function $t \mapsto \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma$ is also continuous on $[0, T)$. Therefore, since $(u(0), v(0)) \in \mathcal{W}_1$ and $(u(t_1), v(t_1)) \in \mathcal{W}_2$, then it follows from the definition of $\mathcal{W}_1$ and $\mathcal{W}_2$ that there exists $s \in (0, t_1)$ such that

$$\|u(s)\|^2_{1,\Omega} + \|v(s)\|^2_{1,\Omega} = (p+1)\int_\Omega F(u(s), v(s))\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u(s))\mathrm{d}\Gamma. \tag{3.6}$$

As a result, we may define t^* as the supreme of all $s \in (0, t_1)$ satisfying (3.6). Clearly, $t^* \in (0, t_1)$, t^* satisfies (3.6), and $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in (t^*, t_1]$.

We have two cases to consider:

Case 1: $(u(t^*), v(t^*)) \neq (0,0)$. In this case, since t^* satisfies (3.6), we see that $(u(t^*), v(t^*)) \in \mathcal{N}$, the Nehari manifold given in (2.13). Thus, by Lemma 2.6, it follows that $J(u(t^*), v(t^*)) \geq d$. Since $E(t) \geq J(u(t), v(t))$ for all $t \in [0,T)$, one has $E(t^*) \geq d$, which contradicts (3.3).

Case 2: $(u(t^*), v(t^*)) = (0,0)$. Since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in (t^*, t_1]$, then by (2.7) and the definition of $\mathcal{W}_2$, we obtain

$$\begin{aligned}\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 &< C\left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} + |\gamma u(t)|_{k+1}^{k+1}\right)\\ &\leq C\left(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1}\right), \quad \text{for all } t \in (t^*, t_1].\end{aligned}$$

Therefore,

$$\|(u(t), v(t))\|_X^2 < C\left(\|(u(t), v(t))\|_X^{p+1} + \|(u(t), v(t))\|_X^{k+1}\right), \quad \text{for all } t \in (t^*, t_1],$$

which yields,

$$1 < C\left(\|(u(t), v(t))\|_X^{p-1} + \|(u(t), v(t))\|_X^{k-1}\right), \quad \text{for all } t \in (t^*, t_1].$$

It follows that $\|(u(t), v(t))\|_X > s_1$, for all $t \in (t^*, t_1]$, where $s_1 > 0$ is the unique positive solution of the equation $C(s^{p-1} + s^{k-1}) = 1$, where p, $k > 1$. Employing the continuity of the weak solution $(u(t), v(t))$, we obtain that

$$\|(u(t^*), v(t^*))\|_X \geq s_1 > 0,$$

which contradicts the assumption $(u(t^*), v(t^*)) = (0,0)$. Hence, $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0,T)$.

Step 2. We show the weak solution $(u(t), v(t))$ is global solution. By (3.3), we know $J(u(t), v(t)) < d$ for all $t \in [0,T)$, that is,

$$\frac{1}{2}\left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) - \int_\Omega F(u(t), v(t))\mathrm{d}x - \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma < d, \text{ on } [0,T). \tag{3.7}$$

Since $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0,T)$, one has

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \geq c\left(\int_\Omega F(u(t), v(t))\mathrm{d}x + \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right), \text{ on } [0,T), \tag{3.8}$$

where $c = \min\{p+1, k+1\} > 2$. Combining (3.7) and (3.8) yields

$$\int_\Omega F(u(t), v(t))\mathrm{d}x + \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma < \frac{2d}{c-2}, \text{ for all } t \in [0,T). \tag{3.9}$$

By using the energy identity (3.1) and (3.9), we deduce

$$\begin{aligned}&\mathscr{E}(t) + \int_0^t\int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma\mathrm{d}\tau\\ &= E(0) + \int_\Omega F(u(t), v(t))\mathrm{d}x + \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\\ &< d + \frac{2d}{c-2} = d\frac{c}{c-2}, \quad \text{for all } t \in [0,T).\end{aligned} \tag{3.10}$$

By virtue of the monotonicity of g_1, g_2, and g, inequality (2.25) follows. Consequently, by a standard continuation argument, we conclude that the weak solution $(u(t), v(t))$ is indeed a global solutions and it can be extended to $[0, \infty)$.

It remains to show inequality (2.26). Obviously, $E(t) \leq \mathscr{E}(t)$ since $F(u, v)$ and $H(s)$ are non-negative functions. On the other hand, by (3.8) and the definition of $E(t)$, one has

$$E(t) \geq \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) + \left(\frac{1}{2} - \frac{1}{c}\right)\left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) \geq \left(1 - \frac{2}{c}\right)\mathscr{E}(t).$$

Thus, the proof of Theorem 2.8 is now complete. □

4. Uniform decay rates of energy

In this section, we study the uniform decay rate of the energy for the global solution furnished by Theorem 2.8. More precisely, we shall prove Theorem 2.9.

We begin by introducing several functions. Let φ_j, $\varphi : [0, \infty) \to [0, \infty)$ be continuous, increasing, concave functions, vanishing at the origin, and such that

$$\varphi_j(g_j(s)s) \geq |g_j(s)|^2 + s^2 \text{ for } |s| < 1, \quad j = 1, 2; \tag{4.1}$$

and

$$\varphi(g(s)s) \geq |g(s)|^2 \quad \text{for } |s| < 1. \tag{4.2}$$

We also define the function $\Phi : [0, \infty) \to [0, \infty)$ by

$$\Phi(s) := \varphi_1(s) + \varphi_2(s) + \varphi(s) + s, \; s \geq 0. \tag{4.3}$$

We note here that the concave functions φ_1, φ_2, and φ mentioned in (4.1)–(4.2) can always be constructed. To see this, recall the damping g_1, g_2, and g are monotone increasing functions passing through the origin. If g_1, g_2, and g are bounded above and below by linear or superlinear functions near the origin, that is, for all $|s| < 1$,

$$c_1|s|^m \leq |g_1(s)| \leq c_2|s|^m, \; c_3|s|^r \leq |g_2(s)| \leq c_4|s|^r, \; c_5|s|^q \leq |g(s)| \leq c_6|s|^q, \tag{4.4}$$

where m, r, $q \geq 1$, and $c_j > 0$, $j = 1, \ldots, 6$, then we can select

$$\varphi_1(s) = c_1^{-\frac{2}{m+1}}(1 + c_2^2)s^{\frac{2}{m+1}}, \;\; \varphi_2(s) = c_3^{-\frac{2}{r+1}}(1 + c_4^2)s^{\frac{2}{r+1}}, \;\; \varphi = c_5^{-\frac{2}{q+1}}c_6^2 s^{\frac{2}{q+1}}. \tag{4.5}$$

It is straightforward to see the functions in (4.5) verify (4.1)–(4.2). To see this, consider φ_1 for example:

$$\begin{aligned}\varphi_1(g_1(s)s) &= c_1^{-\frac{2}{m+1}}\left(1 + c_2^2\right)(g_1(s)s)^{\frac{2}{m+1}} \geq c_1^{-\frac{2}{m+1}}\left(1 + c_2^2\right)\left(c_1|s|^{m+1}\right)^{\frac{2}{m+1}} \\ &= \left(1 + c_2^2\right)s^2 \geq s^2 + (c_2|s|^m)^2 \geq s^2 + |g_1(s)|^2, \text{ for all } |s| < 1.\end{aligned}$$

In particular, we note that, if g_1, g_2, and g are all linearly bounded near the origin, then (4.5) shows φ_1, φ_2, and φ are all linear functions.

However, if the damping are bounded by sublinear functions near the origin, namely, for all $|s| < 1$,

$$c_1|s|^{\theta_1} \leq |g_1(s)| \leq c_2|s|^{\theta_1}, \; c_3|s|^{\theta_2} \leq |g_2(s)| \leq c_4|s|^{\theta_2}, \; c_5|s|^{\theta} \leq |g(s)| \leq c_6|s|^{\theta}, \tag{4.6}$$

where $0 < \theta_1$, θ_2, $\theta < 1$, and $c_j > 0$, $j = 1, \ldots, 6$, then instead we can select

$$\varphi_1(s) = c_1^{-\frac{2\theta_1}{\theta_1+1}}(1 + c_2^2)s^{\frac{2\theta_1}{\theta_1+1}}, \;\; \varphi_2(s) = c_3^{-\frac{2\theta_2}{\theta_2+1}}(1 + c_4^2)s^{\frac{2\theta_2}{\theta_2+1}}, \;\; \varphi = c_5^{-\frac{2\theta}{\theta+1}}c_6^2 s^{\frac{2\theta}{\theta+1}}. \tag{4.7}$$

In sum, by (4.5) and (4.7), there exist constants C_1, C_2, $C_3 > 0$ such that

$$\varphi_1(s) = C_1 s^{z_1}, \quad \varphi_2(s) = C_2 s^{z_2}, \quad \varphi(s) = C_3 s^{z}, \tag{4.8}$$

where

$$z_1 := \frac{2}{m+1} \text{ or } \frac{2\theta_1}{\theta_1+1}, \quad z_2 := \frac{2}{r+1} \text{ or } \frac{2\theta_2}{\theta_2+1}, \quad z := \frac{2}{q+1} \text{ or } \frac{2\theta}{\theta+1} \tag{4.9}$$

depending on the growth rates of g_1, g_2, and g near the origin, which are specified in (4.4) and (4.6).

Now, we define

$$j := \max\left\{\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z}\right\}. \tag{4.10}$$

It is important to note that $j > 1$ if at least one of g_1, g_2, and g are not linearly bounded near the origin, and in this case, we put

$$\beta := \frac{1}{j-1} > 0. \tag{4.11}$$

For the sake of simplifying the notations, we define

$$\mathbf{D}(t) := \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t] \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma\mathrm{d}\tau.$$

We note here that $\mathbf{D}(t) \geq 0$, by the monotonicity of g_1, g_2, and g, and the energy identity (3.1) can be written as

$$E(t) + \mathbf{D}(t) = E(0). \tag{4.12}$$

For the remainder of the proof of Theorem 2.9, we define

$$T_0 := \max\left\{1, \frac{1}{|\Omega|}, \frac{1}{|\Gamma|}, 8c_0\left(\frac{c}{c-2}\right)\right\} \tag{4.13}$$

where c_0 is the constant in the Poincaré–Wirtinger type of inequality (2.1), and $c = \min\{p+1, k+1\} > 2$.

4.1. Perturbed stabilization estimate

Proposition 4.1. *In addition to Assumptions 1.1 and 2.4, assume that* $1 < p < 5$, $1 < k < 3$, $u_0 \in L^{m+1}(\Omega)$, $v_0 \in L^{r+1}(\Omega)$, $\gamma u_0 \in L^{q+1}(\Gamma)$, $(u_0, v_0) \in \mathcal{W}_1$, *and* $E(0) < d$. *We further assume that* $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ *if* $m > 5$, $v \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(r-1)}(\Omega))$ *if* $r > 5$, *and* $\gamma u \in L^\infty(\mathbb{R}^+; L^{2(q-1)}(\Gamma))$ *if* $q > 3$, *where* (u, v) *is the global solution of (1.1) furnished by Theorem 2.8. Then,*

$$E(T) \leq \hat{C}\left[\Phi(\mathbf{D}(T)) + \int_0^T (\|u(t)\|_2^2 + \|v(t)\|_2^2)\mathrm{d}t\right], \tag{4.14}$$

for all $T \geq T_0$, *where* T_0 *is defined in (4.13),* Φ *is given in (4.3), and* $\hat{C} > 0$ *is independent of* T.

Proof. Let $T \geq T_0$ be fixed. We begin by verifying $u \in L^{m+1}(\Omega \times (0, T))$ for all $T \in [0, \infty)$. Since both u and $u_t \in C([0, T]; L^2(\Omega))$, we can write

$$\begin{aligned} \int_0^T \int_\Omega |u|^{m+1} \mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega \left|\int_0^t u_t(\tau)\mathrm{d}\tau + u_0\right|^{m+1} \mathrm{d}x\mathrm{d}t \\ &\leq 2^m \left(T^{m+1} \|u_t\|_{L^{m+1}(\Omega\times(0,T))}^{m+1} + T\|u_0\|_{m+1}^{m+1}\right) < \infty, \end{aligned}$$

where we have used the regularity enjoyed by u, namely, $u_t \in L^{m+1}(\Omega \times (0, T))$, and the assumption $u_0 \in L^{m+1}(\Omega)$. Note, if $m \leq 5$, then $u_0 \in L^{m+1}(\Omega)$ is not an extra assumption since $u_0 \in H^1(\Omega) \hookrightarrow L^6(\Omega)$.

Similarly, we can show $v \in L^{r+1}(\Omega \times (0,T))$ and $\gamma u \in L^{q+1}(\Gamma \times (0,T))$. It follows that u and v enjoy, respectively, the regularity restrictions imposed on the test function ϕ and ψ, as stated in Definition 2.1. Consequently, we can replace ϕ by u in (2.2) and ψ by v in (2.3), and then the sum of two equations gives

$$\begin{aligned}
&\left[\int_\Omega (u_t u + v_t v)\mathrm{d}x\right]_0^T - \int_0^T \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{d}t + \int_0^T \left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right)\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega (g_1(u_t)u + g_2(v_t)v)\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Gamma g(\gamma u_t)\gamma u \mathrm{d}\Gamma \mathrm{d}t \\
&\quad = \int_0^T\int_\Omega [f_1(u,v)u + f_2(u,v)v]\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Gamma h(\gamma u)\gamma u \mathrm{d}\Gamma\mathrm{d}t.
\end{aligned} \tag{4.15}$$

After a rearrangement of (4.15) and employing the identity (2.6), we obtain

$$\begin{aligned}
2\int_0^T \mathscr{E}(t)\mathrm{d}t =& 2\int_0^T \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{d}t - \left[\int_\Omega (u_t u + v_t v)\mathrm{d}x\right]_0^T \\
&- \int_0^T\int_\Omega (g_1(u_t)u + g_2(v_t)v)\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Gamma g(\gamma u_t)\gamma u\mathrm{d}\Gamma\mathrm{d}t \\
&+ (p+1)\int_0^T\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}t + (k+1)\int_0^T\int_\Gamma H(\gamma u)\mathrm{d}\Gamma\mathrm{d}t.
\end{aligned} \tag{4.16}$$

By recalling (2.7), one has

$$\begin{aligned}
\int_0^T \mathscr{E}(t)\mathrm{d}t \le& \int_0^T \left(\|u_t\|_2^2 + \|v_t\|_2^2\right)\mathrm{d}t + \left|\left[\int_\Omega (u_t u + v_t v)\mathrm{d}x\right]_0^T\right| \\
&+ \left[\int_0^T\int_\Omega |g_1(u_t)u + g_2(v_t)v|\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Gamma |g(\gamma u_t)\gamma u|\mathrm{d}\Gamma\mathrm{d}t\right] \\
&+ C\int_0^T \left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1}\right)\mathrm{d}t.
\end{aligned} \tag{4.17}$$

Now, we start with estimating each term on the right-hand side of (4.17).

1. **Estimate for**

$$\left|\left[\int_\Omega (u_t u + v_t v)\mathrm{d}x\right]_0^T\right|.$$

Notice

$$\begin{aligned}\Big|\int_\Omega (u_t(t)u(t)+v_t(t)v(t))\mathrm{d}x\Big| &\leq \|u_t(t)\|_2\,\|u(t)\|_2+\|v_t(t)\|_2\,\|v(t)\|_2\\ &\leq \frac{1}{2}\left(\|u_t(t)\|_2^2+\|u(t)\|_2^2+\|v_t(t)\|_2^2+\|v(t)\|_2^2\right)\leq c_0\mathscr{E}(t), \quad \text{for all } t\geq 0,\end{aligned}$$

where $c_0>0$ is the constant in the Poincaré–Wirtinger type of inequality (2.1). Thus, by (2.26) and (4.12), it follows that

$$\begin{aligned}\left|\left[\int_\Omega (u_tu+v_tv)\mathrm{d}x\right]_0^T\right| &\leq c_0(\mathscr{E}(T)+\mathscr{E}(0))\leq c_0\left(\frac{c}{c-2}\right)(E(T)+E(0))\\ &\leq c_0\left(\frac{c}{c-2}\right)(2E(T)+\mathbf{D}(T)).\end{aligned} \tag{4.18}$$

2. **Estimate for**

$$\int_0^T\left(\|u\|_{p+1}^{p+1}+\|v\|_{p+1}^{p+1}+|\gamma u|_{k+1}^{k+1}\right)\mathrm{d}t.$$

Since $p<5$, then by the Sobolev Imbedding Theorem, $H^{1-\delta}(\Omega)\hookrightarrow L^{p+1}(\Omega)$, for sufficiently small $\delta>0$, and by using a standard interpolation, we obtain

$$\|u\|_{p+1}\leq C\,\|u\|_{H^{1-\delta}(\Omega)}\leq C\,\|u\|_{1,\Omega}^{1-\delta}\,\|u\|_2^{\delta}.$$

Applying Young's inequality yields

$$\|u\|_{p+1}^{p+1}\leq C\,\|u\|_{1,\Omega}^{(1-\delta)(p+1)}\,\|u\|_2^{\delta(p+1)}\leq \epsilon_0\,\|u\|_{1,\Omega}^{\frac{2(1-\delta)(p+1)}{2-\delta(p+1)}}+C_{\epsilon_0}\,\|u\|_2^2 \tag{4.19}$$

for all $\epsilon_0>0$, and where we have required $\delta<\frac{2}{p+1}$. By (2.26) and (3.3), one has

$$\|u\|_{1,\Omega}^2\leq 2\mathscr{E}(t)\leq\left(\frac{2c}{c-2}\right)E(t)\leq\left(\frac{2c}{c-2}\right)E(0). \tag{4.20}$$

Since $p>1$ and $\delta<\frac{2}{p+1}$, then $\frac{2(1-\delta)(p+1)}{2-\delta(p+1)}>2$, and thus combining (4.19) and (4.20) implies

$$\|u\|_{p+1}^{p+1}\leq\epsilon_0 C(E(0))\,\|u\|_{1,\Omega}^2+C_{\epsilon_0}\,\|u\|_2^2. \tag{4.21}$$

For each $\epsilon>0$, if we choose $\epsilon_0=\frac{\epsilon}{C(E(0))}$, then (4.21) gives

$$\|u\|_{p+1}^{p+1}\leq\epsilon\,\|u\|_{1,\Omega}^2+C(\epsilon,E(0))\,\|u\|_2^2. \tag{4.22}$$

Replacing u by v in (4.19)–(4.22) yields

$$\|v\|_{p+1}^{p+1}\leq\epsilon\,\|v\|_{1,\Omega}^2+C(\epsilon,E(0))\,\|v\|_2^2. \tag{4.23}$$

Also, since $k<3$, then by the Sobolev Imbedding Theorem $|\gamma u|_{k+1}\leq C\,\|u\|_{H^{1-\delta}(\Omega)}$, for sufficiently small $\delta>0$. By employing similar estimates as in (4.19)–(4.22), we deduce

$$|\gamma u|_{k+1}^{k+1}\leq\epsilon\,\|u\|_{1,\Omega}^2+C(\epsilon,E(0))\,\|u\|_2^2. \tag{4.24}$$

A combination of the estimates (4.22)–(4.24) yields

$$\int_0^T \left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1}\right) \mathrm{d}t$$
$$\leq 4\epsilon \int_0^T \mathscr{E}(t)\mathrm{d}t + C(\epsilon, E(0)) \int_0^T \left(\|u\|_2^2 + \|v\|_2^2\right) \mathrm{d}t. \tag{4.25}$$

3. **Estimate for**

$$\int_0^T \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \mathrm{d}t.$$

We introduce the sets:

$$A := \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| < 1\}$$
$$B := \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| \geq 1\}.$$

By Assumption 1.1, we know $g_1(s)s \geq a_1|s|^{m+1} \geq a_1|s|^2$ for $|s| \geq 1$. Therefore, applying (4.1) and the fact φ_1 is concave and increasing implies,

$$\begin{aligned}\int_0^T \|u_t\|_2^2 \,\mathrm{d}t &= \int_A |u_t|^2 \mathrm{d}x\mathrm{d}t + \int_B |u_t|^2 \mathrm{d}x\mathrm{d}t \\ &\leq \int_A \varphi_1(g_1(u_t)u_t)\mathrm{d}x\mathrm{d}t + \int_B g_1(u_t)u_t\mathrm{d}x\mathrm{d}t \\ &\leq T|\Omega|\varphi_1\left(\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t\right) + \int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{4.26}$$

where we have used Jensen's inequality and our choice of T, namely $T|\Omega| \geq 1$. Likewise, one has

$$\int_0^T \|v_t\|_2^2 \,\mathrm{d}t \leq T|\Omega|\varphi_2\left(\int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t\right) + \int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t. \tag{4.27}$$

4. **Estimate for**

$$\int_0^T\int_\Omega |g_1(u_t)u + g_2(v_t)v|\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Gamma |g(\gamma u_t)\gamma u|\mathrm{d}\Gamma\mathrm{d}t.$$

Case 1: m, $r \le 5$ and $q \le 3$.
We will concentrate on evaluating $\int_0^T \int_\Omega |g_1(u_t)u| \mathrm{d}x\mathrm{d}t$. Notice

$$\begin{aligned}\int_0^T \int_\Omega |g_1(u_t)u| \mathrm{d}x\mathrm{d}t &= \int_A |g_1(u_t)u| \mathrm{d}x\mathrm{d}t + \int_B |g_1(u_t)u| \mathrm{d}x\mathrm{d}t \\ &\le \left(\int_0^T \|u\|_2^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_A |g_1(u_t)|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} + \int_B |g_1(u_t)u| \mathrm{d}x\mathrm{d}t \\ &\le \epsilon \int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon \int_A |g_1(u_t)|^2 \mathrm{d}x\mathrm{d}t + \int_B |g_1(u_t)u| \mathrm{d}x\mathrm{d}t \qquad (4.28)\end{aligned}$$

where we have used Hölder's and Young's inequalities. By (4.1), Jensen's inequality and the fact $T|\Omega| \ge 1$, we have

$$\int_A |g_1(u_t)|^2 \mathrm{d}x\mathrm{d}t \le \int_A \varphi_1(g_1(u_t)u_t)\mathrm{d}x\mathrm{d}t \le T|\Omega|\varphi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t \mathrm{d}x\mathrm{d}t\right). \qquad (4.29)$$

Next, we estimate the last term on the right-hand side of (4.28). Since $m \le 5$, then by Assumption 1.1, we know $|g_1(s)| \le b_1|s|^m \le b_1|s|^5$ for $|s| \ge 1$. Therefore, by Hölder's inequality, we deduce

$$\begin{aligned}\int_B |g_1(u_t)u| \mathrm{d}x\mathrm{d}t &\le \left(\int_B |u|^6 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)|^{\frac{6}{5}} \mathrm{d}x\mathrm{d}t\right)^{\frac{5}{6}} \\ &\le \left(\int_0^T \|u\|_6^6 \,\mathrm{d}t\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)||g_1(u_t)|^{\frac{1}{5}} \mathrm{d}x\mathrm{d}t\right)^{\frac{5}{6}} \\ &\le b_1^{\frac{1}{6}} \left(\int_0^T \|u\|_6^6 \,\mathrm{d}t\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)||u_t| \mathrm{d}x\mathrm{d}t\right)^{\frac{5}{6}}. \qquad (4.30)\end{aligned}$$

By recalling inequality (2.25) which states $\mathscr{E}(t) \le d\left(\frac{c}{c-2}\right)$, for all $t \ge 0$, we have

$$\int_0^T \|u\|_6^6 \,\mathrm{d}t \le C \int_0^T \|u\|_{1,\Omega}^6 \,\mathrm{d}t \le C \int_0^T \mathscr{E}(t)^3 \mathrm{d}t \le C \int_0^T \mathscr{E}(t)\mathrm{d}t. \qquad (4.31)$$

Combining (4.30) and (4.31) yields

$$\begin{aligned}\int_B |g_1(u_t)u| \mathrm{d}x\mathrm{d}t &\le C \left(\int_0^T \mathscr{E}(t)\mathrm{d}t\right)^{\frac{1}{6}} \left(\int_0^T \int_\Omega g_1(u_t)u_t \mathrm{d}x\mathrm{d}t\right)^{\frac{5}{6}} \\ &\le \epsilon \int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon \int_0^T \int_\Omega g_1(u_t)u_t \mathrm{d}x\mathrm{d}t \qquad (4.32)\end{aligned}$$

where we have used Young's inequality.

By applying the estimates (4.29) and (4.32), we obtain from (4.28) that

$$\int_0^T\int_\Omega |g_1(u_t)u|\mathrm{d}x\mathrm{d}t \leq 2\epsilon\int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon T|\Omega|\varphi_1\left(\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t, \quad \text{if } m\leq 5. \tag{4.33}$$

Similarly,

$$\int_0^T\int_\Omega |g_2(v_t)v|\mathrm{d}x\mathrm{d}t \leq 2\epsilon\int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon T|\Omega|\varphi_2\left(\int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t, \quad \text{if } r\leq 5. \tag{4.34}$$

Likewise, since $T|\Gamma|\geq 1$, we similarly derive

$$\int_0^T\int_\Gamma |g(\gamma u_t)\gamma u|\mathrm{d}\Gamma\mathrm{d}t \leq 2\epsilon\int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon T|\Gamma|\varphi\left(\int_0^T\int_\Gamma g(\gamma u_t)\gamma u_t\mathrm{d}\Gamma\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Gamma g(\gamma u_t)\gamma u_t\mathrm{d}\Gamma\mathrm{d}t, \ \text{if } q\leq 3. \tag{4.35}$$

<u>*Case 2*</u>: $\max\{m,r\}>5$ or $q>3$.
In this case, we impose the additional assumption $u\in L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))$ if $m>5$, $v\in L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(r-1)}(\Omega))$ if $r>5$, and $\gamma u\in L^\infty(\mathbb{R}^+;L^{2(q-1)}(\Gamma))$ if $q>3$.
We evaluate the last term on the right-hand side of (4.28) for the case $m>5$. By Hölder's inequality, we have

$$\int_B |g_1(u_t)u|\mathrm{d}x\mathrm{d}t \leq \left[\int_B |g_1(u_t)|^{\frac{m+1}{m}}\mathrm{d}x\mathrm{d}t\right]^{\frac{m}{m+1}}\left[\int_B |u|^{m+1}\mathrm{d}x\mathrm{d}t\right]^{\frac{1}{m+1}}. \tag{4.36}$$

Since $|g_1(s)|\leq b_1|s|^m$ for all $|s|\geq 1$, one has

$$\int_B |g_1(u_t)|^{\frac{m+1}{m}}\mathrm{d}x\mathrm{d}t = \int_B |g_1(u_t)||g_1(u_t)|^{\frac{1}{m}}\mathrm{d}x\mathrm{d}t \leq b_1^{\frac{1}{m}}\int_B |g_1(u_t)||u_t|\mathrm{d}x\mathrm{d}t. \tag{4.37}$$

We evaluate the last term in (4.36) using Hölder's inequality:

$$\int_B |u|^{m+1}\mathrm{d}x\mathrm{d}t \leq \int_0^T\int_\Omega |u|^2|u|^{m-1}\mathrm{d}x\mathrm{d}t \leq \int_0^T \|u\|_6^2\,\|u\|_{\frac{3}{2}(m-1)}^{m-1}\,\mathrm{d}t \leq C\,\|u\|_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}^{m-1}\int_0^T \mathscr{E}(t)\mathrm{d}t. \tag{4.38}$$

Now, combining (4.36)–(4.38) yields

$$\int_B |g_1(u_t)u|\mathrm{d}x\mathrm{d}t$$
$$\leq C\,\|u\|^{\frac{m-1}{m+1}}_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}\left(\int_0^T \mathscr{E}(t)\mathrm{d}t\right)^{\frac{1}{m+1}}\left(\int_B |g_1(u_t)||u_t|\mathrm{d}x\mathrm{d}t\right)^{\frac{m}{m+1}}$$
$$\leq \epsilon\,\|u\|^{m-1}_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}\int_0^T \mathscr{E}(t)\mathrm{d}t + C_\epsilon\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t \tag{4.39}$$

where we have used Young's inequality.
By (4.28), (4.29) and (4.39), one has

$$\int_0^T\int_\Omega |g_1(u_t)u|\mathrm{d}x\mathrm{d}t \leq \epsilon\left(1+\|u\|^{m-1}_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}\right)\int_0^T \mathscr{E}(t)\mathrm{d}t$$
$$+ C_\epsilon T|\Omega|\varphi_1\left(\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Omega g_1(u_t)u_t\mathrm{d}x\mathrm{d}t, \quad \text{if } m>5. \tag{4.40}$$

Similarly, we can deduce

$$\int_0^T\int_\Omega |g_2(v_t)v|\mathrm{d}x\mathrm{d}t \leq \epsilon\left(1+\|v\|^{r-1}_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(r-1)}(\Omega))}\right)\int_0^T \mathscr{E}(t)\mathrm{d}t$$
$$+ C_\epsilon T|\Omega|\varphi_2\left(\int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Omega g_2(v_t)v_t\mathrm{d}x\mathrm{d}t, \quad \text{if } r>5; \tag{4.41}$$

and

$$\int_0^T\int_\Gamma |g(\gamma u_t)\gamma u|\mathrm{d}x\mathrm{d}t \leq \epsilon\left(1+\|\gamma u\|^{q-1}_{L^\infty(\mathbb{R}^+;L^{2(q-1)}(\Gamma))}\right)\int_0^T \mathscr{E}(t)\mathrm{d}t$$
$$+ C_\epsilon T|\Gamma|\varphi\left(\int_0^T\int_\Gamma g(\gamma u_t)\gamma u_t\mathrm{d}\Gamma\mathrm{d}t\right) + C_\epsilon\int_0^T\int_\Gamma g(\gamma u_t)\gamma u_t\mathrm{d}\Gamma\mathrm{d}t, \quad \text{if } q>3. \tag{4.42}$$

Now, if we combine the estimates (4.17), (4.18), (4.25)–(4.27), (4.33)–(4.35), (4.40)–(4.42), then by selecting ϵ sufficiently small and since $T \geq T_0 \geq 1$, we conclude

$$\frac{1}{2}\int_0^T \mathscr{E}(t)\mathrm{d}t \leq c_0\left(\frac{c}{c-2}\right)(2E(T)+\mathbf{D}(T)) + C(\epsilon, E(0))\int_0^T\left(\|u\|_2^2+\|v\|_2^2\right)\mathrm{d}t$$
$$+ T\cdot C(\epsilon,|\Omega|,|\Gamma|)\Phi(\mathbf{D}(T)). \tag{4.43}$$

Since $\mathscr{E}(t) \geq E(t)$ for all $t \geq 0$ and $E(t)$ is non-increasing, one has

$$\int_0^T \mathscr{E}(t)\mathrm{d}t \geq \int_0^T E(t)\mathrm{d}t \geq TE(T). \tag{4.44}$$

Appealing to the fact $T \geq T_0 \geq 8c_0 \left(\frac{c}{c-2}\right)$, then (4.43) and (4.44) yield

$$\frac{1}{4}TE(T) \leq c_0 \left(\frac{c}{c-2}\right) \mathbf{D}(T) + C(\epsilon, E(0)) \int_0^T \left(\|u\|_2^2 + \|v\|_2^2\right) \mathrm{d}t \\ + T \cdot C(\epsilon, |\Omega|, |\Gamma|)\Phi(\mathbf{D}(T)). \tag{4.45}$$

Since $T \geq 1$, dividing both sides of (4.45) by T yields

$$\frac{1}{4}E(T) \leq c_0 \left(\frac{c}{c-2}\right) \mathbf{D}(T) + C(\epsilon, E(0)) \int_0^T \left(\|u\|_2^2 + \|v\|_2^2\right) \mathrm{d}t \\ + C(\epsilon, |\Omega|, |\Gamma|)\Phi(\mathbf{D}(T)). \tag{4.46}$$

Finally, if we put $\hat{C} := 4[c_0 \left(\frac{c}{c-2}\right) + C(\epsilon, |\Omega|, |\Gamma|) + C(\epsilon, E(0))]$, then (4.46) shows

$$E(T) \leq \hat{C} \left[\Phi(\mathbf{D}(T)) + \int_0^T \left(\|u(t)\|_2^2 + \|v(t)\|_2^2\right) \mathrm{d}t\right] \tag{4.47}$$

for all $T \geq T_0 = \max\{1, \frac{1}{|\Omega|}, \frac{1}{|\Gamma|}, 8c_0(\frac{c}{c-2})\}$. $\square$

4.2. Explicit approximation of the "good" part $\mathcal{W}_1$ of the potential well

In order to estimate the lower order terms $\int_0^T (\|u(t)\|_2^2 + \|v(t)\|_2^2)\mathrm{d}t$ in (4.14), we shall construct an explicit subset $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$, which approximates the "good" part of the well $\mathcal{W}_1$. By the definition of $J(u, v)$ in (2.9) and the bounds in (2.7), it follows that

$$J(u, v) \geq \frac{1}{2}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - M\left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1}\right).$$

By recalling the constants defined in (2.28), we have

$$\begin{aligned} J(u, v) &\geq \frac{1}{2}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - MR_1\left(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1}\right) - MR_2 \|u\|_{1,\Omega}^{k+1} \\ &\geq \frac{1}{2}\|(u, v)\|_X^2 - MR_1 \|(u, v)\|_X^{p+1} - MR_2 \|(u, v)\|_X^{k+1} \end{aligned} \tag{4.48}$$

where $X = H^1(\Omega) \times H_0^1(\Omega)$.

By recalling the function $\mathcal{G}(s)$ defined in (2.27), namely

$$\mathcal{G}(s) := \frac{1}{2}s^2 - MR_1 s^{p+1} - MR_2 s^{k+1},$$

then inequality (4.48) is equivalent to

$$J(u, v) \geq \mathcal{G}(\|(u, v)\|_X). \tag{4.49}$$

Since p, $k > 1$, then

$$\mathcal{G}'(s) = s\left(1 - MR_1(p+1)s^{p-1} - MR_2(k+1)s^{k-1}\right)$$

has only one positive zero at, say at $s_0 > 0$, where s_0 satisfies:

$$MR_1(p+1)s_0^{p-1} + MR_2(k+1)s_0^{k-1} = 1. \tag{4.50}$$

It is easy to verify that $\sup_{s\in[0,\infty)} \mathcal{G}(s) = \mathcal{G}(s_0) > 0$. Thus, we can define the following set as in (2.29):

$$\tilde{\mathcal{W}}_1 := \{(u,v) \in X : \|(u,v)\|_X < s_0,\ J(u,v) < \mathcal{G}(s_0)\}.$$

It is important to note $\tilde{\mathcal{W}}_1$ is not a trivial set. In fact, for any $(u,v) \in X$, there exists a scalar $\epsilon > 0$ such that $\epsilon(u,v) \in \tilde{\mathcal{W}}_1$. Moreover, we have the following result.

Proposition 4.2. *$\tilde{\mathcal{W}}_1$ is a subset of $\mathcal{W}_1$.*

Proof. We first show $\mathcal{G}(s_0) \le d$. Fix $(u,v) \in X \setminus \{(0,0)\}$, then (4.49) yields $J(\lambda(u,v)) \ge \mathcal{G}(\lambda \|(u,v)\|_X)$ for all $\lambda \ge 0$. It follows that

$$\sup_{\lambda\ge 0} J(\lambda(u,v)) \ge \mathcal{G}(s_0).$$

Therefore, by Lemma 2.7, one has

$$d = \inf_{(u,v)\in X\setminus\{(0,0)\}} \sup_{\lambda\ge 0} J(\lambda(u,v)) \ge \mathcal{G}(s_0). \tag{4.51}$$

Moreover, for all $\|(u,v)\|_X < s_0$, by employing (2.7) and (2.28), we argue

$$\begin{aligned}
&(p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma \\
&\le (p+1)MR_1\left(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1}\right) + (k+1)MR_2\,\|u\|_{1,\Omega}^{k+1} \\
&\le \|(u,v)\|_X^2\left[(p+1)MR_1\,\|(u,v)\|_X^{p-1} + (k+1)MR_2\,\|(u,v)\|_X^{k-1}\right] \\
&< \|(u,v)\|_X^2\left[(p+1)MR_1 s_0^{p-1} + (k+1)MR_2 s_0^{k-1}\right] \\
&= \|(u,v)\|_X^2 = \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2
\end{aligned} \tag{4.52}$$

where we have used (4.50). Therefore, by the definition of $\mathcal{W}_1$, it follows that $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$. □

For each fixed sufficiently small $\delta > 0$, we can define a closed subset of $\tilde{\mathcal{W}}_1$ as in (2.30), namely,

$$\tilde{\mathcal{W}}_1^\delta := \{(u,v) \in X : \|(u,v)\|_X \le s_0 - \delta,\ J(u,v) \le \mathcal{G}(s_0-\delta)\},$$

and we show $\tilde{\mathcal{W}}_1^\delta$ is invariant under the dynamics.

Proposition 4.3. *Assume $\delta > 0$ is sufficiently small and $E(0) \le \mathcal{G}(s_0-\delta)$. If (u,v) is the global solution of (1.1) furnished by Theorem 2.8 and $(u_0,v_0) \in \tilde{\mathcal{W}}_1^\delta$, then $(u(t),v(t)) \in \tilde{\mathcal{W}}_1^\delta$ for all $t \ge 0$.*

Proof. By the fact $J(u(t),v(t)) \le E(t) \le E(0)$ and by assumption $E(0) \le \mathcal{G}(s_0-\delta)$, we obtain $J(u(t),v(t)) \le \mathcal{G}(s_0-\delta)$ for all $t \ge 0$. To show $\|(u(t),v(t))\|_X \le s_0 - \delta$ for all $t \ge 0$, we argue by contradiction. Since $\|(u_0,v_0)\|_X \le s_0 - \delta$ and $(u,v) \in C(\mathbb{R}^+;X)$, we can assume in contrary that there exists $t_1 > 0$ such that $\|(u(t_1),v(t_1))\|_X = s_0 - \delta + \epsilon$ for some $\epsilon \in (0,\delta)$. Therefore, by (4.49), we obtain that $J((u(t_1),v(t_1))) \ge \mathcal{G}(s_0-\delta+\epsilon) > \mathcal{G}(s_0-\delta)$ since $\mathcal{G}(t)$ is strictly increasing on $(0,s_0)$. However, this contradicts the fact that $J(u(t),v(t)) \le \mathcal{G}(s_0-\delta)$ for all $t \ge 0$. □

4.3. Absorption of the lower order terms

Proposition 4.4. *In addition to Assumptions 1.1 and 2.4, further assume $(u_0,v_0) \in \tilde{\mathcal{W}}_1^\delta$ and $E(0) < \mathcal{G}(s_0-\delta)$ for some $\delta > 0$. If $1 < p < 5$ and $1 < k < 3$, then the global solution (u,v) of the system (1.1) furnished by Theorem 2.8 satisfies the inequality*

$$\int_0^T \left(\|u(t)\|_2^2 + \|v(t)\|_2^2\right) \mathrm{d}t \le C_T \Phi(\mathbf{D}(T)) \tag{4.53}$$

for all $T \ge T_0$, where T_0 is specified in (4.13).

Proof. We follow the standard compactness-uniqueness approach and argue by contradiction.

Step 1: Limit problem from the contradiction hypothesis. Let us fix $T \ge T_0$. Suppose there is a sequence of initial data

$$\{u_0^n, v_0^n, u_1^n, v_1^n\} \subset \mathcal{W}_1^\delta \times (L^2(\Omega))^2$$

such that the corresponding weak solutions (u^n, v^n) verify

$$\lim_{n\to\infty} \frac{\Phi(\mathbf{D}_n(T))}{\int_0^T \left(\|u^n(t)\|_2^2 + \|v^n(t)\|_2^2\right) \mathrm{d}t} = 0, \tag{4.54}$$

where

$$\mathbf{D}_n(T) := \int_0^T \int_\Omega \left[g_1\left(u_t^n\right) u_t^n + g_2\left(v_t^n\right) v_t^n\right] \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Gamma g\left(\gamma u_t^n\right) \gamma u_t^n \mathrm{d}\Gamma\mathrm{d}t.$$

By the energy estimate (2.25), we have $\int_0^T (\|u^n(t)\|_2^2 + \|v^n(t)\|_2^2)\mathrm{d}t \le 2Td(\frac{c}{c-2})$ for all $n \in \mathbb{N}$. Therefore, it follows from (4.54) that

$$\lim_{n\to\infty} \Phi(\mathbf{D}_n(T)) = 0. \tag{4.55}$$

By recalling (4.26)–(4.27) and (4.55), one has

$$\lim_{n\to\infty} \int_0^T \left(\|u_t^n\|_2^2 + \|v_t^n\|_2^2\right) \mathrm{d}t = 0. \tag{4.56}$$

By Assumption 1.1, we know $a_1|s|^{m+1} \le g_1(s)s \le b_1|s|^{m+1}$ for all $|s| \ge 1$, and so

$$|g_1(s)|^{\frac{m+1}{m}} \le b_1^{\frac{m+1}{m}} |s|^{m+1} \le b_1^{\frac{m+1}{m}} \frac{1}{a_1} g_1(s)s, \quad \text{for all } |s| \ge 1. \tag{4.57}$$

In addition, since g_1 is increasing and vanishing at the origin, we know

$$|g_1(s)| \le b_1, \quad \text{for all } |s| < 1. \tag{4.58}$$

If we define the sets

$$\begin{aligned} A_n &:= \{(x,t) \in \Omega \times (0,T) : |u_t^n(x,t)| < 1\} \\ B_n &:= \{(x,t) \in \Omega \times (0,T) : |u_t^n(x,t)| \ge 1\}, \end{aligned} \tag{4.59}$$

then (4.57) and (4.58) imply

$$\begin{aligned} \int_0^T \int_\Omega |g_1\left(u_t^n\right)|^{\frac{m+1}{m}} \mathrm{d}x\mathrm{d}t &= \int_{A_n} |g_1\left(u_t^n\right)|^{\frac{m+1}{m}} \mathrm{d}x\mathrm{d}t + \int_{B_n} |g_1\left(u_t^n\right)|^{\frac{m+1}{m}} \mathrm{d}x\mathrm{d}t \\ &\le b_1^{\frac{m+1}{m}} |\Omega| T + b_1^{\frac{m+1}{m}} \frac{1}{a_1} \int_0^T \int_\Omega g_1\left(u_t^n\right) u_t^n \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.60}$$

Since $\int_0^T\int_\Omega g_1(u_t^n)u_t^n\mathrm{d}x\mathrm{d}t\to 0$, as $n\to\infty$, (implied by (4.55)), then (4.60) shows

$$\sup_{n\in\mathbb{N}}\int_0^T\int_\Omega |g_1(u_t^n)|^{\frac{m+1}{m}}\mathrm{d}x\mathrm{d}t<\infty. \tag{4.61}$$

Note (4.56) implies, on a subsequence, $u_t^n\to 0$ a.e. in $\Omega\times(0,T)$. Thus, $g_1(u_t^n)\to 0$ a.e. in $\Omega\times(0,T)$. Consequently, by (4.61) and the fact $\frac{m+1}{m}>1$, we conclude,

$$g_1(u_t^n)\to 0 \text{ weakly in } L^{\frac{m+1}{m}}(\Omega\times(0,T)). \tag{4.62}$$

Similarly, by following (4.57)–(4.61) step by step, we may deduce

$$\sup_{n\in\mathbb{N}}\int_0^T\int_\Gamma |g(\gamma u_t^n)|^{\frac{q+1}{q}}\mathrm{d}\Gamma\mathrm{d}t<\infty. \tag{4.63}$$

Notice (4.55) shows $\int_0^T\int_\Gamma g(\gamma u_t^n)\gamma u_t^n\mathrm{d}\Gamma\mathrm{d}t\to 0$ as $n\to\infty$. So on a subsequence $g(\gamma u_t^n)\gamma u_t^n\to 0$ a.e. in $\Gamma\times(0,T)$, and since g is increasing and vanishing at the origin, we see $g(\gamma u_t^n)\to 0$ a.e. in $\Gamma\times(0,T)$. Therefore, by (4.63), it follows that

$$g(\gamma u_t^n)\to 0 \text{ weakly in } L^{\frac{q+1}{q}}(\Gamma\times(0,T)). \tag{4.64}$$

Now, notice (2.25) implies that the sequence of quadratic energy $\mathscr{E}_n(t):=\frac{1}{2}(\|u^n\|_{1,\Omega}^2+\|v^n\|_{1,\Omega}^2+\|u_t^n\|_2^2+\|v_t^n\|_2^2)$ is uniformly bounded on $[0,T]$. Therefore, $\{u^n,v^n,u_t^n,v_t^n\}$ is a bounded sequence in $L^\infty(0,T;H^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega))$. So, on a subsequence, we have

$$\begin{aligned} &u^n\longrightarrow u \ \text{ weakly}^* \text{ in } L^\infty(0,T;H^1(\Omega)),\\ &v^n\longrightarrow v \ \text{ weakly}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)). \end{aligned} \tag{4.65}$$

We note here that for any $0<\epsilon\le 1$, the imbedding $H^1(\Omega)\hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, and $H^{1-\epsilon}(\Omega)\hookrightarrow L^2(\Omega)$. Thus, by Aubin's Compactness Theorem, for any $\alpha>1$, there exists a subsequence such that

$$\begin{aligned} &u^n\longrightarrow u \ \text{ strongly in } L^\alpha(0,T;H^{1-\epsilon}(\Omega)),\\ &v^n\longrightarrow v \ \text{ strongly in } L^\alpha(0,T;H_0^{1-\epsilon}(\Omega)). \end{aligned} \tag{4.66}$$

In addition, for any fixed $1\le s<6$, we know $H^{1-\epsilon}(\Omega)\hookrightarrow L^s(\Omega)$ for sufficiently small $\epsilon>0$. Hence, it follows from (4.66) that

$$u^n\longrightarrow u \text{ and } v^n\longrightarrow v \ \text{ strongly in } L^s(\Omega\times(0,T)), \tag{4.67}$$

for any $1\le s<6$. Similarly, by (4.66), one also has

$$\gamma u^n\longrightarrow\gamma u \ \text{ strongly in } L^{s_0}(\Gamma\times(0,T)), \tag{4.68}$$

for any $s_0<4$. Consequently, on a subsequence,

$$\begin{aligned} &u^n\to u \text{ and } v^n\to v \text{ a.e. in } \Omega\times(0,T),\\ &\gamma u^n\to\gamma u \text{ a.e. in } \Gamma\times(0,T). \end{aligned} \tag{4.69}$$

Now let $t\in(0,T)$ be fixed. If $\phi\in C(\overline{\Omega\times(0,t)})$, then by (3.4), we have

$$|f_j(u^n,v^n)\phi|\le C(|u^n|^p+|v^n|^p) \text{ in } \Omega\times(0,t),\quad j=1,2. \tag{4.70}$$

Since $p<5$, using (4.67), (4.69)–(4.70), and the Generalized Dominated Convergence Theorem, we arrive at

$$\lim_{n\to\infty}\int_0^t\int_\Omega f_j(u^n,v^n)\phi\mathrm{d}x\mathrm{d}\tau=\int_0^t\int_\Omega f_j(u,v)\phi\mathrm{d}x\mathrm{d}\tau,\quad j=1,2. \tag{4.71}$$

Similarly, applying (4.68)–(4.69), the assumption $k < 4$ and $|h(s)| \leq C|s|^k$, we may deduce

$$\lim_{n\to\infty} \int_0^t \int_\Gamma h(\gamma u^n)\gamma\phi \mathrm{d}\Gamma \mathrm{d}\tau = \int_0^t \int_\Gamma h(\gamma u)\gamma\phi \mathrm{d}\Gamma \mathrm{d}\tau. \tag{4.72}$$

If we select a test function $\phi \in C(\overline{\Omega \times (0,t)}) \cap C([0,t]; H^1(\Omega))$ such that $\phi(t) = \phi(0) = 0$ and $\phi_t \in L^2(\Omega \times (0,t))$, then (2.2) gives

$$\begin{aligned} &\int_0^t \left[-\left(u_t^n, \phi_t\right)_\Omega + (u^n, \phi)_{1,\Omega}\right] \mathrm{d}\tau + \int_0^t \int_\Omega g_1\left(u_t^n\right)\phi \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma g\left(\gamma u_t^n\right)\gamma\phi \mathrm{d}\Gamma\mathrm{d}\tau \\ &\quad = \int_0^t \int_\Omega f_1(u^n, v^n)\phi \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma h(\gamma u^n)\gamma\phi \mathrm{d}\Gamma\mathrm{d}\tau. \end{aligned} \tag{4.73}$$

By employing (4.56), (4.62), (4.64), (4.65), (4.71)–(4.72), we can pass to the limit in (4.73) to obtain

$$\int_0^t (u, \phi)_{1,\Omega} \mathrm{d}\tau = \int_0^t \int_\Omega f_1(u, v)\phi \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma\phi \mathrm{d}\Gamma\mathrm{d}\tau. \tag{4.74}$$

Now we fix $\tilde{\phi} \in H^1(\Omega) \cap C(\overline{\Omega})$ and substitute $\phi(x,\tau) := \tau(t-\tau)\tilde{\phi}(x)$ into (4.74). Differentiating the result twice with respect to t yields

$$(u(t), \tilde{\phi})_{1,\Omega} = \int_\Omega f_1(u(t), v(t))\tilde{\phi} \mathrm{d}x + \int_\Gamma h(\gamma u(t))\gamma\tilde{\phi} \mathrm{d}\Gamma. \tag{4.75}$$

If we select a sequence $\tilde{\phi}_n \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\tilde{\phi}_n \to u(t)$ in $H^1(\Omega)$, for a fixed t, then $\tilde{\phi}_n \to u(t)$ in $L^6(\Omega)$. Now, since $|f_1(u,v)| \leq C(|u|^p + |v|^p)$ with $p < 5$, $|h(s)| \leq C|s|^k$ with $k < 3$, then by Hölder's inequality, we can pass to the limit as $n \to \infty$ in (4.75) (where $\tilde{\phi}$ is replaced by $\tilde{\phi}_n$), to obtain

$$\|u(t)\|_{1,\Omega}^2 = \int_\Omega f_1(u(t), v(t))u(t) \mathrm{d}x + \int_\Gamma h(\gamma u(t))\gamma u(t) \mathrm{d}\Gamma. \tag{4.76}$$

In addition, by repeating (4.73)–(4.76) for (2.3), we can derive

$$\|v(t)\|_{1,\Omega}^2 = \int_\Omega f_2(u(t), v(t))v(t) \mathrm{d}x. \tag{4.77}$$

Adding (4.76) and (4.77) gives

$$\begin{aligned} \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 = &\int_\Omega (f_1(u(t), v(t))u(t) + f_2(u(t), v(t))v(t)) \mathrm{d}x \\ &+ \int_\Gamma h(\gamma u(t))\gamma u(t) \mathrm{d}\Gamma, \quad \text{for any } t \in (0,T). \end{aligned} \tag{4.78}$$

Next, we show $(u(t), v(t)) \in \tilde{\mathcal{W}}_1^\delta$ a.e. on $[0,T]$. Indeed, by (4.65)–(4.66) and referring to Proposition 2.9 in [28], we obtain, on a subsequence

$$\begin{aligned} &u^n(t) \longrightarrow u(t) \text{ weakly in } H^1(\Omega) \quad \text{a.e. } t \in [0,T]; \\ &v^n(t) \longrightarrow v(t) \text{ weakly in } H_0^1(\Omega) \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{4.79}$$

It follows that

$$\|u(t)\|_{1,\Omega} \leq \liminf_{n\to\infty} \|u^n(t)\|_{1,\Omega} \quad \text{and} \quad \|v(t)\|_{1,\Omega} \leq \liminf_{n\to\infty} \|v^n(t)\|_{1,\Omega}\,, \tag{4.80}$$

for a.e. $t \in [0,T]$. Since the initial data $\{u_0^n, v_0^n\} \in \tilde{\mathcal{W}}_1^\delta$ and $E(0) < \mathcal{G}(s_0 - \delta)$, then Proposition 4.3 shows the corresponding global solutions $\{u^n(t), v^n(t)\} \in \tilde{\mathcal{W}}_1^\delta$ for all $t \geq 0$. Then, by the definition of $\tilde{\mathcal{W}}_1^\delta$ one knows $\|(u^n(t), v^n(t))\|_X \leq s_0 - \delta$, and $J(u^n(t), v^n(t)) \leq \mathcal{G}(s_0 - \delta)$ for all $t \geq 0$. Thus, (4.80) implies $\|(u(t), v(t))\|_X \leq s_0 - \delta$ a.e. on $[0,T]$. In order to show $J(u(t), v(t)) \leq \mathcal{G}(s_0 - \delta)$ a.e. on $[0,T]$, we note that

$$\begin{aligned}\mathcal{G}(s_0 - \delta) &\geq J(u^n(t), v^n(t))\\ &= \frac{1}{2}\left(\|u^n(t)\|_{1,\Omega} + \|v^n(t)\|_{1,\Omega}\right) - \int\limits_\Omega F(u^n(t), v^n(t))\mathrm{d}x - \int\limits_\Gamma H(\gamma u^n(t))\mathrm{d}\Gamma.\end{aligned} \tag{4.81}$$

Since the imbedding $H^1(\Omega) \to H^{1-\epsilon}(\Omega)$ is compact and $p < 5$, $k < 3$, we obtain from (4.79) that

$$\begin{aligned}&u^n(t) \longrightarrow u(t), \;\; v^n(t) \longrightarrow v(t) \;\text{ strongly in }\; L^{p+1}(\Omega), \quad \text{a.e. on } [0,T]\\ &\gamma u^n(t) \longrightarrow \gamma u(t) \;\text{ strongly in }\; L^{k+1}(\Gamma), \quad \text{a.e. on } [0,T].\end{aligned} \tag{4.82}$$

By (2.7), (4.82), and the Generalized Dominated Convergence Theorem, one has, on a subsequence

$$\begin{aligned}&\lim_{n\to\infty} \int\limits_\Omega F(u^n(t), v^n(t))\mathrm{d}x = \int\limits_\Omega F(u(t), v(t))\mathrm{d}x, \quad \text{a.e. on } [0,T],\\ &\lim_{n\to\infty} \int\limits_\Gamma H(\gamma u^n(t))\mathrm{d}\Gamma = \int\limits_\Gamma H(\gamma u(t))\mathrm{d}\Gamma, \quad \text{a.e. on } [0,T].\end{aligned} \tag{4.83}$$

Applying (4.80) and (4.83), we can take the limit inferior on both side of the inequality (4.81) to obtain

$$\mathcal{G}(s_0 - \delta) \geq J(u(t), v(t)), \quad \text{a.e. on } [0,T].$$

Hence $(u(t), v(t)) \in \tilde{\mathcal{W}}_1^\delta \subset \mathcal{W}_1$ a.e. on $[0,T]$. Therefore, by the definition of $\mathcal{W}_1$ and (4.78), necessarily we have $(u(t), v(t)) = (0,0)$ a.e. on $[0,T]$. Therefore, (4.67) implies

$$u^n \longrightarrow 0 \text{ and } v^n \longrightarrow 0 \text{ strongly in } L^s(\Omega \times (0,T)), \quad \text{for any } s < 6. \tag{4.84}$$

Step 2: Re-normalize the sequence $\{u^n, v^n\}$. We define

$$N_n := \left(\int_0^T \left(\|u^n\|_2^2 + \|v^n\|_2^2\right) \mathrm{d}t\right)^{\frac{1}{2}}.$$

By (4.84), one has $u^n \longrightarrow 0$ and $v^n \longrightarrow 0$ in $L^2(\Omega \times (0,T))$, and so, $N_n \longrightarrow 0$ as $n \to \infty$. If we set

$$y^n := \frac{u^n}{N_n} \text{ and } z^n := \frac{v^n}{N_n},$$

then clearly

$$\int_0^T \left(\|y^n\|_2^2 + \|z^n\|_2^2\right) \mathrm{d}t = 1. \tag{4.85}$$

By the contradiction hypothesis (4.54), namely

$$\lim_{n\to\infty} \frac{\Phi(\mathbf{D}_n(T))}{N_n^2} = 0, \tag{4.86}$$

and along with (4.26)–(4.27), we obtain

$$\lim_{n\to\infty} \frac{\int_0^T \left(\|u_t^n\|_2^2 + \|v_t^n\|_2^2 \right) dt}{N_n^2} = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \int_0^T \left(\|y_t^n\|_2^2 + \|z_t^n\|_2^2 \right) dt = 0. \tag{4.87}$$

We next show

$$\frac{g_1(u_t^n)}{N_n} \longrightarrow 0 \ \text{ strongly in } L^{\frac{m+1}{m}}(\Omega \times (0,T)). \tag{4.88}$$

Recall the definition of the sets A_n and B_n in (4.59). Since $N_n \longrightarrow 0$ as $n \to \infty$, we can let n be sufficiently large such that $N_n < 1$, then by using (4.1), (4.57), Hölder's and Jensen's inequalities, we deduce

$$\begin{aligned}
\int_0^T \int_\Omega \left| \frac{g_1(u_t^n)}{N_n} \right|^{\frac{m+1}{m}} dxdt &= \int_{A_n} \left| \frac{g_1(u_t^n)}{N_n} \right|^{\frac{m+1}{m}} dxdt + \int_{B_n} \left| \frac{g_1(u_t^n)}{N_n} \right|^{\frac{m+1}{m}} dxdt \\
&\le C(T, |\Omega|) \left(\int_{A_n} \left| \frac{g_1(u_t^n)}{N_n} \right|^2 dxdt \right)^{\frac{m+1}{2m}} + \frac{1}{N_n^2} \int_{B_n} |g_1(u_t^n)|^{\frac{m+1}{m}} dxdt \\
&\le C(T, |\Omega|) \left(\frac{1}{N_n^2} \int_{A_n} \varphi_1(g_1(u_t^n)u_t^n) dxdt \right)^{\frac{m+1}{2m}} + \frac{b_1^{\frac{m+1}{m}}}{a_1 N_n^2} \int_{B_n} g_1(u_t^n) u_t^n dxdt \\
&\le C(T, |\Omega|) \left(\frac{\Phi(\mathbf{D}_n(T))}{N_n^2} \right)^{\frac{m+1}{2m}} + \frac{b_1^{\frac{m+1}{m}}}{a_1} \frac{\Phi(\mathbf{D}_n(T))}{N_n^2} \longrightarrow 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where we have used (4.86) and the fact $T \ge T_0 \ge \frac{1}{|\Omega|}$. Thus, our desired result (4.88) follows.

Likewise, we can prove

$$\frac{g(\gamma u_t^n)}{N_n} \longrightarrow 0 \ \text{ strongly in } L^{\frac{q+1}{q}}(\Gamma \times (0,T)). \tag{4.89}$$

Let E_n be the total energy corresponding to the solution (u^n, v^n). So (2.26) shows $E_n(t) \ge 0$ for all $t \ge 0$. Also by (4.14) and (4.85)–(4.86), we obtain $\lim_{n\to\infty} \frac{E_n(T)}{N_n^2} \le \hat{C}$, which implies $\{\frac{E_n(T)}{N_n^2}\}$ is uniformly bounded. The energy identity (4.12) shows $E_n(T) + \mathbf{D}_n(T) = E_n(0)$, and thus $\{\frac{E_n(0)}{N_n^2}\}$ is also uniformly bounded. Moreover, since $E_n'(t) \le 0$ for all $t \ge 0$, one has $\{\frac{E_n(t)}{N_n^2}\}$ is uniformly bounded on $[0,T]$, and along with the energy inequality (2.26), we conclude that the sequence

$$\left\{ \frac{\mathscr{E}_n(t)}{N_n^2} = \frac{1}{2} \left(\|y^n\|_{1,\Omega}^2 + \|z^n\|_{1,\Omega}^2 + \|y_t^n\|_2^2 + \|z_t^n\|_2^2 \right) \right\}$$

is uniformly bounded on $[0,T]$, where $\mathscr{E}_n$ is the quadratic energy corresponding to (u^n, v^n). Therefore, $\{y^n, z^n, y_t^n, z_t^n\}$ is a bounded sequence in $L^\infty(0,T; H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega))$. Therefore, on a subsequence,

$$\begin{aligned}
&y^n \longrightarrow y \ \text{ weakly}^* \text{ in } L^\infty(0,T; H^1(\Omega)), \\
&z^n \longrightarrow z \ \text{ weakly}^* \text{ in } L^\infty(0,T; H_0^1(\Omega)).
\end{aligned} \tag{4.90}$$

As in (4.66)–(4.69), we may deduce that, on subsequences

$$y^n \longrightarrow y \text{ and } z^n \longrightarrow z \text{ strongly in } L^s(\Omega \times (0,T)), \tag{4.91}$$

for any $s < 6$, and

$$\gamma y^n \longrightarrow \gamma y \text{ strongly in } L^{s_0}(\Gamma \times (0,T)), \tag{4.92}$$

for any $s_0 < 4$. Note (4.85) and (4.91) show that

$$\lim_{n\to\infty} \int_0^T \left(\|y^n\|_2^2 + \|z^n\|_2^2\right) \mathrm{d}t = \int_0^T \left(\|y\|_2^2 + \|z\|_2^2\right) \mathrm{d}t = 1. \tag{4.93}$$

However, by Hölder's inequality,

$$\begin{aligned}\int_0^T \int_\Omega |y^n||u^n|^{p-1} \mathrm{d}x\mathrm{d}t &\leq \left(\int_0^T \int_\Omega |y^n|^5 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{5}} \left(\int_0^T \int_\Omega |u^n|^{\frac{5}{4}(p-1)} \mathrm{d}x\mathrm{d}t\right)^{\frac{4}{5}} \\ &\longrightarrow \|y\|_{L^5(\Omega\times(0,T))} \cdot 0 = 0\end{aligned} \tag{4.94}$$

where we have used (4.91), (4.84), and the fact $\frac{5}{4}(p-1) < 5$.

Similarly,

$$\lim_{n\to\infty} \int_0^T \int_\Omega |z^n||v^n|^{p-1} \mathrm{d}x\mathrm{d}t = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_0^T \int_\Gamma |\gamma y^n||\gamma u^n|^{k-1} \mathrm{d}\Gamma\mathrm{d}t = 0. \tag{4.95}$$

Since $|f_j(u^n, v^n)| \leq C(|u^n|^p + |v^n|^p)$, $j = 1, 2$, it follows that,

$$\int_0^t \int_\Omega \left|\frac{f_j(u^n, v^n)}{N_n}\phi\right| \mathrm{d}x\mathrm{d}\tau \leq C \int_0^t \int_\Omega (|y^n||u^n|^{p-1} + |z^n||v^n|^{p-1}) \mathrm{d}x\mathrm{d}\tau \longrightarrow 0, \tag{4.96}$$

for any $t \in (0,T)$, $\phi \in C(\overline{\Omega \times (0,t)})$, and where we have used (4.94)–(4.95). Likewise,

$$\int_0^t \int_\Gamma \left|\frac{h(\gamma u^n)}{N_n}\gamma\phi\right| \mathrm{d}\Gamma\mathrm{d}\tau \leq C \int_0^t \int_\Omega |\gamma y^n||\gamma u^n|^{k-1} \mathrm{d}\Gamma\mathrm{d}\tau \longrightarrow 0. \tag{4.97}$$

Dividing both sides of (4.73) by N_n yields

$$\begin{aligned}&\int_0^t [-(y_t^n, \phi_t)_\Omega + (y^n, \phi)_{1,\Omega}]\mathrm{d}\tau + \int_0^t \int_\Omega \frac{g_1(u_t^n)}{N_n}\phi \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma \frac{g(\gamma u_t^n)}{N_n}\gamma\phi \mathrm{d}\Gamma\mathrm{d}\tau \\ &\quad = \int_0^t \int_\Omega \frac{f_1(u^n, v^n)}{N_n}\phi \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma \frac{h(\gamma u^n)}{N_n}\gamma\phi \mathrm{d}\Gamma\mathrm{d}\tau.\end{aligned} \tag{4.98}$$

where $\phi \in C(\overline{\Omega \times (0,t)}) \cap C([0,t]; H^1(\Omega))$ such that $\phi(t) = \phi(0) = 0$ and $\phi_t \in L^2(\Omega \times (0,t))$.

By using (4.87), (4.88)–(4.89), (4.90), and (4.96)–(4.97), we can pass to the limit in (4.98) to find

$$\int_0^t (y^n, \phi)_{1,\Omega} \mathrm{d}\tau = 0, \quad \text{for all } t \in (0,T). \tag{4.99}$$

Now, fix an arbitrary $\tilde{\phi} \in H^1(\Omega) \cap C(\overline{\Omega})$ and substitute $\phi(x,\tau) = \tau(t-\tau)\tilde{\phi}(x)$ into (4.99). Differentiating the result twice yields

$$(y(t), \tilde{\phi})_{1,\Omega} = 0, \quad \text{for all } t \in (0,T), \tag{4.100}$$

which implies $y(t) = 0$ in $H^1(\Omega)$ for all $t \in (0,T)$. Similarly, we can show $z(t) = 0$ in $H_0^1(\Omega)$ for all $t \in (0,T)$. However, this contradicts the fact (4.93). Hence, the proof of Proposition 4.4 is complete. □

Remark 4.5. *We can iterate the estimate (4.53) on time intervals $[mT,(m+1)T]$, $m = 0,1,2,\ldots$, and obtain*

$$\int_{mT}^{(m+1)T} \left(\|u(t)\|_2^2 + \|v(t)\|_2^2\right) dt \leq C_T \Phi(\mathbf{D}(T)), \ \ m = 0,1,2,\ldots \tag{4.101}$$

It is important to note, by the contradiction hypothesis made in the proof of Proposition 4.4, the constant C_T in (4.101) does not depend on m.

4.4. Proof of Theorem 2.9

We are now ready to prove Theorem 2.9: the uniform decay rates of energy.

Proof. Combining Propositions 4.1 and 4.4 yields $E(T) \leq \hat{C}(1+C_T)\Phi(\mathbf{D}(T))$ for all $T \geq T_0$. If we set $\Phi_T = \hat{C}(1+C_T)\Phi$, where C_T is as given in (4.53), then the energy identity (4.12) shows that

$$E(T) \leq \Phi_T(\mathbf{D}(T)) = \Phi_T(E(0) - E(T)),$$

which implies

$$E(T) + \Phi_T^{-1}(E(T)) \leq E(0).$$

By iterating the estimate on intervals $[mT,(m+1)T]$, $m = 0,1,2,\ldots$, we have

$$E((m+1)T) + \Phi_T^{-1}(E((m+1)T)) \leq E(mT), \quad m = 0,1,2,\ldots$$

Therefore, by Lemma 3.3 in [19], one has

$$E(mT) \leq S(m) \quad \text{for all } m = 0,1,2,\ldots \tag{4.102}$$

where S is the solution the ODE:

$$S' + \left[I - \left(I + \Phi_T^{-1}\right)^{-1}\right](S) = 0, \quad S(0) = E(0), \tag{4.103}$$

where I denotes the identity mapping. However, we note that

$$\begin{aligned} I - \left(I+\Phi_T^{-1}\right)^{-1} &= \left(I+\Phi_T^{-1}\right) \circ \left(I+\Phi_T^{-1}\right)^{-1} - \left(I+\Phi_T^{-1}\right)^{-1} = \Phi_T^{-1} \circ \left(I+\Phi_T^{-1}\right)^{-1} \\ &= \Phi_T^{-1} \circ \left(\Phi_T \circ \Phi_T^{-1} + \Phi_T^{-1}\right)^{-1} = \Phi_T^{-1} \circ \Phi_T \circ (I+\Phi_T)^{-1} = (I+\Phi_T)^{-1}. \end{aligned}$$

It follows that the ODE (4.103) can be reduced to:

$$S' + (I+\Phi_T)^{-1}(S) = 0, \quad S(0) = E(0), \tag{4.104}$$

where (4.104) has a unique solutions defined on $[0,\infty)$. Since Φ_T is increasing passing through the origin, we have $(I+\Phi_T)^{-1}$ is also increasing and vanishing at zero. So if we write (4.104) in the form $S' = -(I+\Phi_T)^{-1}(S)$, then it follows that $S(t)$ is decreasing and $S(t) \to 0$ as $t \to \infty$.

For any $t > T$, there exists $m \in \mathbb{N}$ such that $t = mT + \delta$ with $0 \leq \delta < T$, and so $m = \frac{t}{T} - \frac{\delta}{T} > \frac{t}{T} - 1$. By (4.102) and the fact $E(t)$ and $S(t)$ are decreasing, we obtain

$$E(t) = E(mT+\delta) \leq E(mT) \leq S(m) \leq S\left(\frac{t}{T} - 1\right), \quad \text{for any } t > T. \tag{4.105}$$

If g_1, g_2, g are linearly bounded near the origin, then (4.5) shows that φ_1, φ_2, φ are linear, and it follows that Φ_T is linear, which implies $(I+\Phi_T)^{-1}$ is also linear. Therefore, the ODE (4.104) is of the form $S'+w_0 S=0$, $S(0)=E(0)$ (for some positive constant w_0), whose solution is given by: $S(t)=E(0)\mathrm{e}^{-w_0 t}$. Thus, from (4.105) we know

$$E(t)\leq E(0)\mathrm{e}^{-w_0\left(\frac{t}{T}-1\right)}=(\mathrm{e}^{w_0}E(0))\mathrm{e}^{-\frac{w_0}{T}t}$$

for $t>T$. Consequently, if we set $w:=\frac{w_0}{T}$ and choose $\tilde{C}$ sufficiently large, then we conclude

$$E(t)\leq \tilde{C}E(0)\mathrm{e}^{-wt},\ \ t\geq 0,$$

which provides the exponential decay estimate (2.31).

If at least one of g_1, g_2, and g are not linearly bounded near the origin, then we can show the decay of $E(t)$ is algebraic. Indeed, by (4.8) we may choose $\varphi_1(s)=C_1 s^{z_1}$, $\varphi_2(s)=C_2 s^{z_2}$, $\varphi(s)=C_3 s^{z}$, where $0<z_1,z_2,z\leq 1$ are given in (4.9). Also recall that $j:=\max\{\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z}\}>1$, as defined in (4.10). Now, we study the function $(I+\Phi_T)^{-1}$. Notice, if $y=(I+\Phi_T)^{-1}(s)$ for $s\geq 0$, then $y\geq 0$. In addition,

$$\begin{aligned}
s&=(I+\Phi_T)y=y+\hat{C}(1+C_T)(\varphi_1(y)+\varphi_2(y)+\varphi(y)+y)\\
&\leq C(\varphi_1(y)+\varphi_2(y)+\varphi(y)+y)\leq Cy^{\min\{z_1,z_2,z\}},\quad \text{for all }\ 0\leq y\leq 1.
\end{aligned}$$

It follows that there exists $C_0>0$ such that $y\geq C_0 s^j$ for all $0\leq y\leq 1$, that is,

$$(I+\Phi_T)^{-1}(s)\geq C_0 s^j\ \text{ provided }\ 0\leq (I+\Phi_T)^{-1}(s)\leq 1. \tag{4.106}$$

Recall we have pointed out that $S(t)$ is decreasing to zero as $t\to\infty$, so $(I+\Phi_T)^{-1}(S(t))$ is also decreasing to zero as $t\to\infty$. Hence, there exists $t_0\geq 0$ such that $(I+\Phi_T)^{-1}(S(t))\leq 1$, whenever $t\geq t_0$. Therefore, (4.106) implies

$$S'(t)=-(I+\Phi_T)^{-1}(S(t))\leq -C_0 S(t)^j\quad \text{if } t\geq t_0.$$

So, $S(t)\leq \hat{S}(t)$ for all $t\geq t_0$ where $\hat{S}$ is the solution of the ODE

$$\hat{S}'(t)=-C_0\hat{S}(t)^j,\ \ \hat{S}(t_0)=S(t_0). \tag{4.107}$$

Since the solution of (4.107) is

$$\hat{S}(t)=[C_0(j-1)(t-t_0)+S(t_0)^{1-j}]^{-\frac{1}{j-1}}\quad \text{for all }\ t\geq t_0,$$

and along with (4.105), it follows that

$$E(t)\leq S\left(\frac{t}{T}-1\right)\leq \hat{S}\left(\frac{t}{T}-1\right)=\left[C_0(j-1)\left(\frac{t}{T}-1-t_0\right)+S(t_0)^{1-j}\right]^{-\frac{1}{j-1}}$$

for all $t\geq T(t_0+1)$. Since $S(t_0)$ depends on the initial energy $E(0)$, there exists a positive constant $C(E(0))$ depending on $E(0)$ such that

$$E(t)\leq C(E(0))(1+t)^{-\frac{1}{j-1}},\quad \text{for all }\ t\geq 0,$$

where $j>1$. Thus, the proof of Theorem 2.9 is complete. □

5. Blow up of potential well solutions

This section is devoted to prove the blow-up result: Theorem 2.10. We begin by showing $\mathcal{W}_2$ is invariant under the dynamics of (1.1). More precisely, we have the following lemma.

Lemma 5.1. *In addition to Assumptions 1.1 and 2.4, further assume that* $(u_0, v_0) \in \mathcal{W}_2$ *and* $E(0) < d$. *If* $1 < p \leq 5$ *and* $1 < k \leq 3$, *then the weak solution* $(u(t), v(t)) \in \mathcal{W}_2$ *for all* $t \in [0, T)$, *and*

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 > 2 \min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} d, \quad \text{for all } t \in [0, T), \tag{5.1}$$

where $[0, T)$ *is the maximal interval of existence.*

Proof. Since $E(0) < d$, we have shown in the proof of Theorem 2.8 that $(u(t), v(t)) \in \mathcal{W}$ for all $t \in [0, T)$. To show that $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, T)$, we proceed by contradiction. Assume there exists $t_1 \in (0, T)$ such that $(u(t_1), v(t_1)) \notin \mathcal{W}_2$, then it must be $(u(t_1), v(t_1)) \in \mathcal{W}_1$. Recall that the weak solution $(u, v) \in C([0, T); H^1(\Omega) \times H_0^1(\Omega))$, and in the proof of Theorem 2.8, we have shown the continuity of the function

$$t \mapsto (p+1) \int_\Omega F(u(t), v(t))\mathrm{d}t + (k+1) \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma.$$

Since $(u(0), v(0)) \in \mathcal{W}_2$, and $(u(t_1), v(t_1)) \in \mathcal{W}_1$, it follows that there exists $s \in (0, t_1]$ such that

$$\|u(s)\|_{1,\Omega}^2 + \|v(s)\|_{1,\Omega}^2 = (p+1) \int_\Omega F(u(s), v(s))\mathrm{d}x + (k+1) \int_\Gamma H(\gamma u(s))\mathrm{d}\Gamma. \tag{5.2}$$

Now we define t^* as the infinimum of all $s \in (0, t_1]$ satisfying (5.2). By continuity, one has $t^* \in (0, t_1]$ satisfying (5.2), and $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, t^*)$. Thus, we have two cases to consider.

<u>*Case 1*</u>: $(u(t^*), v(t^*)) \neq (0, 0)$. Since t^* satisfies (5.2), it follows $(u(t^*), v(t^*)) \in \mathcal{N}$, and by Lemma 2.6, we know $J(u(t^*), v(t^*)) \geq d$. Thus $E(t^*) \geq d$, contradicting $E(t) \leq E(0) < d$ for all $t \in [0, T)$.

<u>*Case 2*</u>: $(u(t^*), v(t^*)) = (0, 0)$. Since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, t^*)$, by utilizing a similar argument as in the proof of Theorem 2.8, we obtain $\|(u(t), v(t))\|_X > s_1$, for all $t \in [0, t^*)$, where $s_1 > 0$. By the continuity of the weak solution $(u(t), v(t))$, we obtain that $\|(u(t^*), v(t^*))\|_X \geq s_1 > 0$, contradicting the assumption $(u(t^*), v(t^*)) = (0, 0)$. It follows that $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, T)$.

It remains to show inequality (5.1). Let $(u, v) \in \mathcal{W}_2$ be fixed. By recalling (2.20) in Lemma 2.7 which states that the only critical point in $(0, \infty)$ for the function $\lambda \mapsto J(\lambda(u, v))$ is $\lambda_0 > 0$, where λ_0 satisfies the equation

$$\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) = (p+1)\lambda_0^{p-1} \int_\Omega F(u, v)\mathrm{d}x + (k+1)\lambda_0^{k-1} \int_\Gamma H(\gamma u)\mathrm{d}\Gamma. \tag{5.3}$$

Since $(u, v) \in \mathcal{W}_2$, then $\lambda_0 < 1$. In addition, we recall the function $\lambda \mapsto J(\lambda(u, v))$ attains its absolute maximum over the positive axis at its critical point $\lambda = \lambda_0$. Thus, by Lemma 2.7 and (5.3), it follows that

$$\begin{aligned}
d &\leq \sup_{\lambda \geq 0} J(\lambda(u, v)) = J(\lambda_0(u, v)) \\
&= \frac{1}{2}\lambda_0^2 \left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - \lambda_0^{p+1} \int_\Omega F(u, v)\mathrm{d}x - \lambda_0^{k+1} \int_\Gamma H(\gamma u)\mathrm{d}\Gamma \\
&\leq \lambda_0^2 \left[\frac{1}{2}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right) - \min\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right)\right] \\
&= \frac{1}{2}\lambda_0^2 \max\left\{\frac{p-1}{p+1}, \frac{k-1}{k+1}\right\}\left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right).
\end{aligned}$$

Since $\lambda_0 < 1$, one has

$$\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 \geq \frac{2d}{\lambda_0^2} \min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} > 2 \min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} d,$$

completing the proof of Lemma 5.1. □

Now, we prove Theorem 2.10: the blow up of potential well solutions.

Proof. In order to show the maximal existence time T is finite, we argue by contradiction. Assume the weak solution $(u(t), v(t))$ can be extended to $[0, \infty)$, then Lemma 5.1 says $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, \infty)$. Moreover, by the assumption $0 \le E(0) < \rho d$, the energy $E(t)$ remains nonnegative:

$$0 \le E(t) \le E(0) < \rho d \quad \text{for all } \; t \in [0, \infty). \tag{5.4}$$

To see this, assume that $E(t_0) < 0$ for some $t_0 \in (0, \infty)$. Then, the blow-up results in [15] assert that

$$\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty,$$

as $t \to T^-$, for some $0 < T < \infty$, that is, the weak solution $(u(t), v(t))$ must blow up in finite time, which contradicts our assumption.

Now, define

$$N(t) := \|u(t)\|_2^2 + \|v(t)\|_2^2\,,$$
$$S(t) := \int_\Omega F(u(t), v(t))\mathrm{d}x + \int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma \ge 0.$$

Since u_t, $v_t \in C([0, \infty); L^2(\Omega))$, it follows that

$$N'(t) = 2\int_\Omega [u(t)u_t(t) + v(t)v_t(t)]\mathrm{d}x. \tag{5.5}$$

Recall in the proof of Proposition 4.1, we have verified u and v enjoy, respectively, the regularity restrictions imposed on the test function ϕ and ψ, as stated in Definition 2.1. Consequently, we can replace ϕ by u in (2.2) and ψ by v in (2.3) and sum the two equations to obtain:

$$\begin{aligned}\frac{1}{2}N'(t) = &\int_\Omega (u_1u_0 + v_1v_0)\mathrm{d}x + \int_0^t\int_\Omega (|u_t|^2 + |v_t|^2)\mathrm{d}x\mathrm{d}\tau - \int_0^t \left(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2\right)\mathrm{d}\tau \\ &- \int_0^t\int_\Omega (g_1(u_t)u + g_2(v_t)v)\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_\Gamma g(\gamma u_t)\gamma u\mathrm{d}\Gamma\mathrm{d}\tau \\ &+ (p+1)\int_0^t\int_\Omega F(u,v)\mathrm{d}x\mathrm{d}\tau + (k+1)\int_0^t\int_\Gamma H(\gamma u)\mathrm{d}\Gamma\mathrm{d}\tau, \qquad \text{a.e. } [0,\infty),\end{aligned} \tag{5.6}$$

where we have used (2.6). Since $p \le 5$ and $k \le 3$, then by Assumption 1.1, one can check that the RHS of (5.6) is absolutely continuous, and thus, we can differentiate both sides of (5.6) to obtain

$$\begin{aligned}\frac{1}{2}N''(t) = &\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) - \left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) \\ &- \int_\Omega (g_1(u_t)u + g_2(v_t)v)\mathrm{d}x - \int_\Gamma g(\gamma u_t)\gamma u\mathrm{d}\Gamma \\ &+ (p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma, \quad \text{a.e. } [0,\infty).\end{aligned} \tag{5.7}$$

The assumption $|g_1(s)| \le b_1|s|^m$ for all $s \in \mathbb{R}$ implies

$$\begin{aligned}\left|\int_\Omega g_1(u_t(t))u(t)\mathrm{d}x\right| &\le b_1 \int_\Omega |u_t(t)|^m|u(t)|\mathrm{d}x\\ &\le C\,\|u(t)\|_{m+1}\,\|u_t(t)\|_{m+1}^m\\ &\le C\,\|u(t)\|_{p+1}\,\|u_t(t)\|_{m+1}^m,\end{aligned} \tag{5.8}$$

where we have used Hölder's inequality and the assumption $p > m$. In addition, the assumption $F(u, v) \ge \alpha_0(|u|^{p+1} + |v|^{p+1})$ for some $\alpha_0 > 0$ yields

$$\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} \le \frac{1}{\alpha_0}\int_\Omega F(u(t), v(t))\mathrm{d}x \le \frac{1}{\alpha_0}S(t). \tag{5.9}$$

It follows from (5.8)–(5.9) that

$$\left|\int_\Omega g_1(u_t(t))u(t)\mathrm{d}x\right| \le CS(t)^{\frac{1}{p+1}}\,\|u_t(t)\|_{m+1}^m \le \epsilon S(t)^{\frac{m+1}{p+1}} + C_\epsilon\,\|u_t(t)\|_{m+1}^{m+1}, \tag{5.10}$$

where we have used Young's inequality.

Since $p > r$, we may similarly deduce

$$\left|\int_\Omega g_2(v_t(t))v(t)\mathrm{d}x\right| \le \epsilon S(t)^{\frac{r+1}{p+1}} + C_\epsilon\,\|v_t(t)\|_{r+1}^{r+1}. \tag{5.11}$$

In order to estimate $|\int_\Gamma g(\gamma u_t(t))\gamma u(t)\mathrm{d}\Gamma|$, depending on different assumptions on parameters, there are two cases to consider: either $k > q$ or $p > 2q - 1$.

Case 1: $k > q$. In this case, the estimate is straightforward. As in (5.8), we have

$$\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)\mathrm{d}x\right| \le C|\gamma u(t)|_{k+1}|\gamma u_t(t)|_{q+1}^q. \tag{5.12}$$

Since $H(s)$ is homogeneous of order $k+1$ and $H(s) > 0$ for all $s \in \mathbb{R}$, then $H(s) \ge \min\{H(1), H(-1)\}|s|^{k+1}$, where $H(1)$, $H(-1) > 0$. Thus,

$$\int_\Gamma |\gamma u(t)|^{k+1}\mathrm{d}\Gamma \le C\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma \le CS(t). \tag{5.13}$$

It follows from (5.12)–(5.13), Young's inequality, and the assumption $k > q$ that

$$\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)\mathrm{d}x\right| \le CS(t)^{\frac{1}{k+1}}|\gamma u_t(t)|_{q+1}^q \le \epsilon S(t)^{\frac{q+1}{k+1}} + C_\epsilon|\gamma u_t(t)|_{q+1}^{q+1}. \tag{5.14}$$

Case 2: $p > 2q - 1$. We shall employ a useful inequality that was shown in [15], namely,

$$|\gamma u|_{q+1} \le C\left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\right), \tag{5.15}$$

where $\frac{p-1}{2(p-q)} \le \beta < 1$. Indeed, the proof of (5.15) requires careful analysis involving the following trace and interpolation theorems:

- Trace theorem:

$$|\gamma u|_{q+1} \le C\,\|u\|_{W^{s,q+1}(\Omega)}, \quad \text{where } s > \frac{1}{q+1}.$$

- Interpolation theorem (see [32]):

$$W^{1-\theta,r}(\Omega) = [H^1(\Omega), L^{p+1}(\Omega)]_\theta,$$

where $r = \frac{2(p+1)}{(1-\theta)(p+1)+2\theta}$, $\theta \in [0,1]$, and as usual $[\cdot,\cdot]_\theta$ denotes the interpolation bracket.

The reader may refer to [15] for the details of the proof of (5.15).

In addition, since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \geq 0$, one has

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \leq \max\{p+1, k+1\} S(t), \quad \text{for all } t \geq 0. \tag{5.16}$$

Now we apply (5.15) and the assumption $|g(s)| \leq b_3 |s|^q$ to obtain

$$\begin{aligned}\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)\mathrm{d}\Gamma\right| &\leq b_3 \int_\Gamma |\gamma u(t)||\gamma u_t(t)|^q \mathrm{d}\Gamma \leq b_3 |\gamma u(t)|_{q+1} |\gamma u_t(t)|_{q+1}^q \\ &\leq C\left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\right)|\gamma u_t(t)|_{q+1}^q \\ &\leq C S(t)^{\frac{\beta}{q+1}} |\gamma u_t(t)|_{q+1}^q \leq \epsilon S(t)^\beta + C_\epsilon |\gamma u_t(t)|_{q+1}^{q+1}.\end{aligned} \tag{5.17}$$

where we have used (5.16), (5.9), and Young's inequality.

Combining (5.7), (5.10)–(5.11), (5.14), and (5.17) yields

$$\begin{aligned}&\frac{1}{2}N''(t) + C_\epsilon\left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\ &\quad \geq \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) - \left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) \\ &\quad - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right) \\ &\quad + (p+1)\int_\Omega F(u,v)\mathrm{d}x + (k+1)\int_\Gamma H(\gamma u)\mathrm{d}\Gamma, \quad \text{a.e. } t \in [0,\infty),\end{aligned} \tag{5.18}$$

where

$$j_0 := \begin{cases} \frac{q+1}{k+1}, & \text{if } k > q, \\ \beta, & \text{if } p > 2q-1. \end{cases}$$

Since $\beta < 1$, it follows $j_0 < 1$.

Rearranging the terms in the definition (2.8) of the total energy $E(t)$ gives

$$\begin{aligned}-\left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) &= \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) - 2\int_\Omega F(u(t),v(t))\mathrm{d}x \\ &\quad - 2\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma - 2E(t).\end{aligned} \tag{5.19}$$

It follows from (5.18)–(5.19) that

$$\begin{aligned}&\frac{1}{2}N''(t) + C_\epsilon\left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\ &\quad \geq (p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma \\ &\quad - 2E(t) - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right), \quad \text{a.e. } t \in [0,\infty).\end{aligned} \tag{5.20}$$

Since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, \infty)$, then by Lemma 5.1, we deduce

$$\begin{aligned}
&(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma \\
&\quad > \min\left\{\frac{p-1}{p+1},\frac{k-1}{k+1}\right\}\left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\right) \\
&\quad > 2\min\left\{\frac{p-1}{p+1},\frac{k-1}{k+1}\right\}\cdot \min\left\{\frac{p+1}{p-1},\frac{k+1}{k-1}\right\} d = 2\rho d, \qquad (5.21)
\end{aligned}$$

for all $t \in [0, \infty)$, where $\rho \le 1$ is defined in (2.34).

Note (5.4) implies there exists $\delta > 0$ such that

$$0 \le E(t) \le E(0) \le (1-\delta)\rho d \quad \text{for all } t \in [0,\infty). \qquad (5.22)$$

Combining (5.20)–(5.22) yields

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon\left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\quad > \delta\left[(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right] + 2(1-\delta)\rho d \\
&\qquad - 2E(t) - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right) \\
&\quad \ge \delta\left[(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right] \\
&\qquad - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right), \quad \text{a.e. } t \in [0,\infty). \qquad (5.23)
\end{aligned}$$

Now, we consider two cases: $S(t) > 1$ and $S(t) \le 1$.

If $S(t) > 1$, then since $p > \max\{m, r\}$ and $j_0 < 1$, one has $S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \le 3S(t)$. In this case, we choose $0 < \epsilon \le \frac{1}{6}\delta\min\{p-1, k-1\}$, and thus, (5.23) and the definition of $S(t)$ imply

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon\left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\quad \ge \delta\left[(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right] - 3\epsilon S(t) \\
&\quad \ge \frac{1}{2}\delta\left[(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right] > \delta\rho d, \qquad (5.24)
\end{aligned}$$

for a.e. $t \in [0, \infty)$, where the inequality (5.21) has been used.

If $S(t) \le 1$, then $S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \le 3$. In this case, we choose $0 < \epsilon \le \frac{1}{3}\delta\rho d$. Thus, it follows from (5.23) and (5.21) that

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon\left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\quad \ge \delta\left[(p-1)\int_\Omega F(u(t),v(t))\mathrm{d}x + (k-1)\int_\Gamma H(\gamma u(t))\mathrm{d}\Gamma\right] - 3\epsilon \\
&\quad > 2\delta\rho d - 3\epsilon \ge \delta\rho d, \quad \text{a.e. } t \in [0,\infty). \qquad (5.25)
\end{aligned}$$

Therefore, if we choose $\epsilon \leq \frac{1}{6}\delta \min\{p-1, k-1, 2\rho d\}$, then it follows from (5.24)–(5.25) that

$$N''(t) + 2C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) > 2\delta\rho d, \quad \text{a.e. } t \in [0,\infty). \tag{5.26}$$

Integrating (5.26) yields

$$N'(t) - N'(0) + 2C_\epsilon \int_0^t \left(\|u_t(\tau)\|_{m+1}^{m+1} + \|v_t(\tau)\|_{r+1}^{r+1} + |\gamma u_t(\tau)|_{q+1}^{q+1}\right) \mathrm{d}\tau \geq (2\delta\rho d)t, \tag{5.27}$$

for all $t \in [0,\infty)$.

By the restrictions on damping in (2.33), one has

$$\begin{aligned}
&\int_0^t \left(\|u_t(\tau)\|_{m+1}^{m+1} + \|v_t(\tau)\|_{r+1}^{r+1} + |\gamma u_t(\tau)|_{q+1}^{q+1}\right) \mathrm{d}\tau \\
&\quad \leq C \left(\int_0^t \int_\Omega (g_1(u_t)u_t + g_2(v_t)v_t)\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t \mathrm{d}\Gamma \mathrm{d}\tau\right) \\
&\quad = C(E(0) - E(t)) < C\rho d \leq Cd, \quad \text{for all } t \in [0,\infty),
\end{aligned} \tag{5.28}$$

where we have used the energy identity (3.1) and the energy estimate (5.4).

A combination of (5.27) and (5.28) yields

$$N'(t) \geq (2\delta\rho d)t + N'(0) - C(\epsilon)d, \quad \text{for all } t \in [0,\infty). \tag{5.29}$$

Integrating (5.29) yields

$$N(t) \geq (\delta\rho d)t^2 + [N'(0) - C(\epsilon)d]t + N(0), \quad \text{for all } t \in [0,\infty). \tag{5.30}$$

It is important to note here (5.30) asserts $N(t)$ has a *quadratic* growth rate as $t \to \infty$.

On the other hand, we can estimate $N(t)$ directly as follows. Note,

$$\begin{aligned}
\|u(t)\|_2^2 &= \int_\Omega \left| u_0 + \int_0^t u_t(\tau)\mathrm{d}\tau \right|^2 \mathrm{d}x \\
&\leq 2\|u_0\|_2^2 + 2t\left(\int_0^t \int_\Omega |u_t(\tau)|^2 \mathrm{d}x\mathrm{d}\tau\right) \\
&\leq 2\|u_0\|_2^2 + Ct^{1+\frac{m-1}{m+1}}\left(\int_0^t \int_\Omega |u_t(\tau)|^{m+1}\mathrm{d}x\mathrm{d}\tau\right)^{\frac{2}{m+1}} \\
&\leq 2\|u_0\|_2^2 + Cd^{\frac{2}{m+1}}t^{\frac{2m}{m+1}}, \quad \text{for all } t \in [0,\infty)
\end{aligned}$$

where we have used (5.28). Likewise,

$$\|v(t)\|_2^2 \leq 2\|v_0\|_2^2 + Cd^{\frac{2}{r+1}}t^{\frac{2r}{r+1}}, \quad \text{for all } t \in [0,\infty).$$

It follows that

$$N(t) \leq 2\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) + C(d^{\frac{2}{m+1}}t^{\frac{2m}{m+1}} + d^{\frac{2}{r+1}}t^{\frac{2r}{r+1}}), \quad \text{for all } t \in [0,\infty). \tag{5.31}$$

Since $\frac{2m}{m+1} < 2$ and $\frac{2r}{r+1} < 2$, then (5.31) contradicts the quadratic growth of $N(t)$, as $t \to \infty$. Therefore, we conclude that weak solution $(u(t), v(t))$ cannot be extended to $[0,\infty)$, and thus, it must be the case that there exists $t_0 \in (0,\infty)$ such that $E(t_0) < 0$. Hence, the proof of Theorem 2.10 is complete. □

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(Received: February 16, 2012; revised: July 20, 2012)