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Global strong solutions for the three-dimensional Hasegawa-Mima model with partial dissipation

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We study the three-dimensional Hasegawa-Mima model of turbulent magnetized plasma with horizontal viscous terms and a weak vertical dissipative term. In particular, we establish the global existence and uniqueness of strong solutions for this model. *Published by AIP Publishing.* https://doi.org/10.1063/1.5022099

I. INTRODUCTION

A. Literature

In 1977, Hasegawa and Mima introduced a system in Refs. 8 and 9 to elucidate the drift wave turbulence in Tokamak, the most advanced magnetic confinement device. The three-dimensional inviscid Hasegawa-Mima equations can be written as (cf. Refs. 2, 4, 8, 9, 17, and 22)

$$\frac{\partial w}{\partial t} + J(\phi, w) + \frac{\partial \phi}{\partial z} = 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}(\Delta_h \phi - \phi) + J(\phi, \Delta_h \phi) + \gamma \frac{\partial \phi}{\partial y} - \frac{\partial w}{\partial z} = 0, \tag{1.2}$$

where $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ is the Jacobian and $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplacian. System (1.1) and (1.2) describes the coupling of the drift modes to the ion-acoustic waves that propagate along the magnetic field. Here, ϕ is the electrostatic potential and simultaneously is the stream function for the horizontal flow in the xy-plane. Moreover, w represents the normalized ion velocity in the z-direction, and γ is a constant which is proportional to the density gradient.

Like the three-dimensional Euler equations of inviscid incompressible fluid, the only conserved quantity for the 3D Hasegawa-Mima equations (1.1) and (1.2) is the kinetic energy, and the global regularity problem is open. Nevertheless, by adding the full viscosity to (1.1) and (1.2), Zhang and Guo[22] proved the global regularity and the existence of global attractors for a viscous and forced 3D Hasegawa-Mima model using standard tools from the theory of Navier-Stokes equations. On the other hand, Cao, Farhat, and Titi[4] proposed and studied an inviscid three-dimensional modified version of (1.1) and (1.2), the pseudo-Hasegawa-Mima equations,

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla_h w - U_0 L \frac{\partial \omega}{\partial z} = 0, \tag{1.3}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla_h \omega - \frac{U_0}{L}\frac{\partial w}{\partial z} = 0, \tag{1.4}$$

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with $\nabla_h \cdot \mathbf{u} = 0$, for some constant U_0, where $\mathbf{u} = (u,v)^{tr}$ is the horizontal component of the velocity vector field $(u,v,w)^{tr}$, and $\omega = \nabla_h \times \mathbf{u}$ is the vorticity. The operator $\nabla_h = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^{tr}$ is the horizontal gradient. In particular, the global well-posedness of the weak solutions to (1.3) and (1.4) was established in Ref. 4. Observe that ω in (1.3) and (1.4) plays the role of the term $\Delta_h \phi - \phi$ in (1.1) and (1.2). Therefore, system (1.3) and (1.4) is a modified version of the Hasegawa-Mima equations (1.1) and (1.2), with the essential difference that the term $\frac{\partial \phi}{\partial z}$ is replace by $\frac{\partial \omega}{\partial z}$. Nevertheless, model (1.3) and (1.4) is simpler than (1.1) and (1.2) in the sense that it has a nice mathematical structure. Indeed, adding and subtracting (1.3) and (1.4) yield a three-dimensional coupled transport system with collinear transport velocities in opposite directions leading to an intensified shear in the vertical direction, which results in exponential growth in the relevant estimates for (1.3) and (1.4) in Ref. 4.

It is worth mentioning other interesting models describing plasma turbulence. For instance, Hasegawa and Wakatani proposed equations for a two-fluid model which describes the resistive drift wave turbulence in Tokamak (cf. Refs. 10 and 11). The existence and uniqueness of strong solutions to the Hasegawa-Wakatani equations have been established by Kondo and Tani.[14]

In the context of geophysical fluid dynamics, there are certain models resembling the structure of Hasegawa-Mima equations (1.1) and (1.2). In particular, Charney[5] and Obukhov[18] derived the following two-dimensional shallow water model from the Euler equations with free surface under a quasi-geostrophic velocity field assumption,

$$\frac{\partial}{\partial t}(\Delta_h \phi_0 - F\phi_0) + J(\phi_0, \Delta_h \phi_0) + J(\phi_0, \phi_B + \beta y) = 0. \tag{1.5}$$

Here $\phi_0(x, y)$ is the amplitude of the surface perturbation at the lowest order in the Rossby number, and the equation $z = \phi_B(x, y)$ describes the given bottom topography. F is the Froude number. One may refer to Ref. 20 for a derivation of model (1.5). For the simple case when ϕ_B is a constant representing a flat bottom, (1.5) reduces to the Hasegawa-Mima-Charney-Obukhov equation,

$$\frac{\partial}{\partial t}(\Delta_h \phi_0 - F\phi_0) + J(\phi_0, \Delta_h \phi_0) + \beta \frac{\partial \phi_0}{\partial x} = 0. \tag{1.6}$$

Since (1.6) bears a close resemblance to the two-dimensional Euler equations, the standard tools for handling the 2D Euler equations can be adopted to analyze (1.6). Indeed, Guo and Han[7] proved the global existence and uniqueness of solutions for (1.6). For other results concerning (1.6), see, e.g., Paumond[19] and Gao and Zhu.[6]

In addition, one may refer to the monographs[16,20] as well as the papers 12, 13, and 21 for other relevant geophysical models.

B. The model

Motivated by the Hasegawa-Mima equations and the Charney-Obukhov equations mentioned in Subsection I A, we introduce and study in this paper the following three-dimensional Hasegawa-Mima model with horizontal viscous terms and a weak vertical dissipative term,

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla_h w - \frac{\partial \psi}{\partial z} = \frac{1}{Re}\Delta_h w, \tag{1.7}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla_h \omega - \frac{\partial w}{\partial z} = \frac{1}{Re}\Delta_h \omega + \epsilon^2 \frac{\partial^2 \psi}{\partial z^2}, \tag{1.8}$$

$$\nabla_h \cdot \mathbf{u} = 0. \tag{1.9}$$

The velocity vector field $(u,v,w)^{tr}$ defined in $\Omega = [0, L]^2 \times [0, 1]$ satisfies the periodic boundary condition with the horizontal velocity $\mathbf{u} = (u,v)^{tr}$. The stream function ψ for the horizontal flow is defined as $\psi = (-\Delta_h)^{-1}\omega$ with $\int_{[0,L]^2} \psi dxdy = 0$, and $\omega = \nabla_h \times \mathbf{u}$. We denote $\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{tr}$ and $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The constant Re is the Reynolds number.

System (1.7)–(1.9) bears a resemblance as the three-dimensional Hasegawa-Mima equations (1.1) and (1.2) with the difference that Hasegawa-Mima equations are inviscid, whereas model (1.7)–(1.9) is regularized by the horizontal viscosity and a partial vertical dissipation. The purpose of introducing and investigating (1.7)–(1.9) is to shed light on the analysis of the inviscid Hasegawa-Mima equations (1.1) and (1.2).

Mathematically, the difficulty of establishing the global regularity for system (1.7)–(1.9) lies in the following aspects:

(i) The physical domain is three-dimensional.
(ii) The regularizing viscosity acts only on the horizontal variables.
(iii) The system contains the troublesome term $\frac{\partial \psi}{\partial z}$.

Since the lack of the viscosity in the vertical direction provides great challenge for establishing the global regularity, we impose a weak dissipative term $\epsilon^2 \frac{\partial^2 \psi}{\partial z^2}$ in Eq. (1.8). Since $\psi = (-\Delta_h)^{-1}\omega$, we remark that, as a dissipation, $\frac{\partial^2 \psi}{\partial z^2}$ is weaker than the vertical viscosity $\frac{\partial^2 \omega}{\partial z^2}$. In *a priori* estimates conducted in Sec. II, the dissipative term $\epsilon^2 \frac{\partial^2 \psi}{\partial z^2}$ plays a vital role in controlling the terms $-\frac{\partial w}{\partial z}$ and $-\frac{\partial \psi}{\partial z}$ with the help of an anisotropic Ladyzhenskaya type inequality (see Lemma 2.1).

C. Preliminaries

In this subsection, we introduce some preliminaries that will be used later in our analysis. Recall the three-dimensional periodic space domain $\Omega = [0, L]^2 \times [0, 1]$. Throughout, the norm for the $L^p(\Omega)$ space, for $p \in [1, \infty]$, is denoted by $\|f\|_p$. The inner product of f and g in the $L^2(\Omega)$ space is denoted by $(f, g) = \int_\Omega fg\,dxdydz$. As usual, the Sobolev space $H^1(\Omega) = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$. In addition, we define the following Hilbert space:

$$H_h^1(\Omega) = \left\{f \in L^2(\Omega) : \nabla_h f \in L^2(\Omega)\right\} \tag{1.10}$$

that features the inner product $(f, g)_{H_h^1(\Omega)} = (f, g) + (\nabla_h f, \nabla_h g)$.

For sufficiently smooth functions f, g, and $\mathbf{u}$, with $\nabla_h \cdot \mathbf{u} = 0$, integration by parts yields

$$(\mathbf{u} \cdot \nabla_h f, g) = -(\mathbf{u} \cdot \nabla_h g, f), \tag{1.11}$$

which immediately implies that

$$(\mathbf{u} \cdot \nabla_h f, f) = 0. \tag{1.12}$$

Recall that the horizontal velocity $\mathbf{u}$, the vertical vorticity ω, and the stream function ψ for the horizontal flow have the following relations:

$$\omega = \nabla_h \times \mathbf{u} = v_x - u_y, \quad \omega = -\Delta_h \psi, \quad \mathbf{u} = (\psi_y, -\psi_x)^{tr}, \tag{1.13}$$

where $\int_{[0,L]^2} \psi\,dxdy = 0$. It follows that if $\omega \in L^2(\Omega)$, then

$$(\omega, \psi) = \|\mathbf{u}\|_2^2. \tag{1.14}$$

In addition, for sufficiently smooth functions f, $\mathbf{u}$, and ψ such that $\mathbf{u} = (\psi_y, -\psi_x)^{tr}$, observe that $\mathbf{u} \cdot \nabla_h \psi = 0$, then apply (1.11) to deduce

$$(\mathbf{u} \cdot \nabla_h f, \psi) = -(\mathbf{u} \cdot \nabla_h \psi, f) = 0. \tag{1.15}$$

D. Main results

Before we state the main result of the paper, we give the definition of a strong solution for system (1.7)–(1.9).

Definition 1.1. We call $(\mathbf{u}, w)^{tr} = (u, v, w)^{tr}$ a *strong solution* on $[0, T]$ for system (1.7)–(1.9) if

(i) $(\mathbf{u}, w)^{tr}$ has the following regularity:

$$\begin{cases} \mathbf{u},\ w \in L^\infty(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \\ \Delta_h \mathbf{u},\ \Delta_h w,\ \omega_z,\ \nabla_h w_z,\ \psi_{zz} \in L^2(\Omega \times (0, T)). \\ \mathbf{u}_t,\ w_t \in L^2(\Omega \times (0, T)), \end{cases} \tag{1.16}$$

(ii) equations (1.17) and (1.18) hold in the following sense:

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla_h w - \frac{\partial \psi}{\partial z} = \frac{1}{Re} \Delta_h w, \text{ in } L^2(\Omega \times (0, T)), \tag{1.17}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla_h \omega - \frac{\partial w}{\partial z} = \frac{1}{Re} \Delta_h \omega + \epsilon^2 \frac{\partial^2 \psi}{\partial z^2}, \text{ in } L^2(0, T; H_h^1(\Omega)'), \tag{1.18}$$

with $\nabla_h \cdot \mathbf{u} = 0$, where $\omega = \nabla_h \times \mathbf{u}$, $\psi = (-\Delta_h)^{-1}\omega$ with $\int_{[0,L]^2} \psi dxdy = 0$, and $(H_h^1(\Omega))'$ is the dual of the space $H_h^1(\Omega)$, defined in (1.10).

Now we are ready to state the main result of the paper: the global existence, uniqueness, and continuous dependence on initial data of strong solutions for our model (1.7)–(1.9).

Theorem 1.2. *Let $T > 0$. Assume $(\mathbf{u}_0, w_0)^{tr} \in (H^1(\Omega))^3$ such that $\nabla_h \cdot \mathbf{u}_0 = 0$, $\int_{[0,L]^2} \mathbf{u}_0 dxdy = 0$, and $\int_{[0,L]^2} w_0 dxdy = 0$. Then system (1.7)–(1.9) admits a unique strong solution $(\mathbf{u}, w)^{tr}$ on $[0, T]$ in the sense of Definition 1.1 satisfying the initial condition $(\mathbf{u}(0), w(0))^{tr} = (\mathbf{u}_0, w_0)^{tr}$. Moreover, the energy equality is valid for every $t \in [0, T]$,*

$$\frac{1}{2}\left(\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2\right) + \int_0^t \left[\frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h \mathbf{u}\|_2^2\right) + \epsilon^2 \|\psi_z\|_2^2\right] ds$$
$$= \frac{1}{2}\left(\|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2\right). \tag{1.19}$$

In addition, the $H^1(\Omega)$ norm of the solution $(\mathbf{u}, w)^{tr}$ has a uniform bound independent of T. That is,

$$\sup_{0 \le t \le T} \left(\|\mathbf{u}(t)\|_{H^1(\Omega)}^2 + \|w(t)\|_{H^1(\Omega)}^2\right) \le K,$$

where K is independent of T but depends only on Re, ϵ, L, $\|\mathbf{u}_0\|_{H^1(\Omega)}$, and $\|w_0\|_{H^1(\Omega)}$. Furthermore, if $\{(\mathbf{u}_0^n, w_0^n)^{tr}\}$ is a bounded sequence of initial data in $H^1(\Omega)$ such that $(\mathbf{u}_0^n, w_0^n)^{tr} \to (\mathbf{u}_0, w_0)^{tr}$ in $L^2(\Omega)$, then the corresponding strong solutions $(\mathbf{u}^n, w^n)^{tr}$ and $(\mathbf{u}, w)^{tr}$ satisfy $(\mathbf{u}^n, w^n)^{tr} \to (\mathbf{u}, w)^{tr}$ in $C([0, T]; L^2(\Omega))$.

II. *A PRIORI* ESTIMATES

In this section, we assume that system (1.7)–(1.9) holds for smooth functions and we establish the following formal *a priori* estimates. However, as we will show in Sec. III, these formal estimates can be justified rigorously by establishing them first for the Galerkin approximation system and then passing to the limit using the appropriate Aubin compactness theorem.

A. Estimate for $\|w\|_2^2 + \|u\|_2^2$

Taking the $L^2(\Omega)$ inner product of the system (1.7) and (1.8) with $(w,\psi)^{tr}$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2 + \|\mathbf{u}\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h \mathbf{u}\|_2^2\right) + \epsilon^2 \|\psi_z\|_2^2 = 0, \tag{2.1}$$

where we have used identities (1.12), (1.14), and (1.15). Integrating (2.1) over the interval $[0, t]$ yields

$$\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2 + \int_0^t \left(\frac{2}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h \mathbf{u}\|_2^2\right) + 2\epsilon^2\|\psi_z\|_2^2\right)ds = \|w_0\|_2^2 + \|\mathbf{u}_0\|_2^2. \tag{2.2}$$

B. Estimate for $\|\omega\|_2^2$

Taking the inner product of (1.8) with ω yields

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \frac{1}{Re}\|\nabla_h\omega\|_2^2 + \epsilon^2\|\mathbf{u}_z\|_2^2 = (w_z, \omega), \tag{2.3}$$

where (1.12) and (1.14) have been used. Thanks to (1.13), we have

$$\begin{aligned}(w_z, \omega) &= \int_\Omega w_z(-\Delta_h\psi)dxdydz = -\int_\Omega \nabla_h w \cdot \nabla_h\psi_z dxdydz \\ &\le \|\nabla_h w\|_2\|\nabla_h\psi_z\|_2 = \|\nabla_h w\|_2\|\mathbf{u}_z\|_2 \le \frac{\epsilon^2}{2}\|\mathbf{u}_z\|_2^2 + \frac{1}{2\epsilon^2}\|\nabla_h w\|_2^2. \end{aligned}\tag{2.4}$$

Combining (2.3) and (2.4) implies

$$\frac{d}{dt}\|\omega\|_2^2 + \frac{2}{Re}\|\nabla_h\omega\|_2^2 + \epsilon^2\|\mathbf{u}_z\|_2^2 \le \frac{1}{\epsilon^2}\|\nabla_h w\|_2^2. \tag{2.5}$$

By integrating (2.5) over the interval $[0, t]$, we obtain

$$\begin{aligned}&\|\omega(t)\|_2^2 + \int_0^t \left(\frac{2}{Re}\|\nabla_h\omega\|_2^2 + \epsilon^2\|\mathbf{u}_z\|_2^2\right)ds \le \|\omega_0\|_2^2 + \frac{1}{\epsilon^2}\int_0^t \|\nabla_h w\|_2^2 ds \\ &\le \|\omega_0\|_2^2 + \frac{Re}{2\epsilon^2}\left(\|\omega_0\|_2^2 + \|\mathbf{u}_0\|_2^2\right),\end{aligned}\tag{2.6}$$

where the last inequality is due to (2.2).

C. An anisotropic Ladyzhenskaya type inequality

We state here the following anisotropic Ladyzhenskaya type inequality which will be useful in subsequent *a priori* estimates. It is worth mentioning that similar inequalities can be found in Ref. 3. However, for the sake of completeness, we present the proof of this technical lemma in the Appendix.

Lemma 2.1. Let $f \in H^1(\Omega)$, $g \in H_h^1(\Omega)$, *and* $h \in L^2(\Omega)$. *Then*

$$\int_\Omega |fgh|dxdydz \le C\left(\|f\|_2 + \|\nabla_h f\|_2\right)^{\frac{1}{2}}\left(\|f\|_2 + \|f_z\|_2\right)^{\frac{1}{2}}\|g\|_2^{\frac{1}{2}}\left(\|g\|_2 + \|\nabla_h g\|_2\right)^{\frac{1}{2}}\|h\|_2.$$

D. Estimate for $\|\nabla_h w\|_2^2$

Taking the inner product of (1.7) with $-\Delta_h w$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla_h w\|_2^2 + \frac{1}{Re}\|\Delta_h w\|_2^2 \\ &\le \int_\Omega |\mathbf{u}\cdot\nabla_h w\Delta_h w|dxdydz + \|\psi_z\|_2\|\Delta_h w\|_2 \\ &\le C\|\omega\|_2^{1/2}(\|\mathbf{u}\|_2 + \|\mathbf{u}_z\|_2)^{1/2}\|\nabla_h w\|_2^{1/2}\|\Delta_h w\|_2^{3/2} + \|\psi_z\|_2\|\Delta_h w\|_2,\end{aligned}$$

where we have used Lemma 2.1.

By employing Young's inequality, we obtain

$$\frac{d}{dt}\|\nabla_h w\|_2^2 + \frac{1}{Re}\|\Delta_h w\|_2^2 \le C\|\omega\|_2^2(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2)\|\nabla_h w\|_2^2 + C\|\psi_z\|_2^2.$$

Thanks to Gronwall's inequality, we have

$$\|\nabla_h w(t)\|_2^2 + \frac{1}{Re}\int_0^t \|\Delta_h w\|_2^2 ds \le C\left(\|\nabla_h w_0\|_2^2 + \int_0^t \|\psi_z\|_2^2 ds\right) e^{\int_0^t C\|\omega\|_2^2(\|\mathbf{u}\|_2^2+\|\mathbf{u}_z\|_2^2)ds}$$
$$\le C(\|\nabla_h w_0\|_2, \|\omega_0\|_2). \tag{2.7}$$

The uniform bound (2.7) is due to estimates (2.2) and (2.6).

E. Estimate for $\|w_z\|_2^2 + \|u_z\|_2^2$

We take the $L^2(\Omega)$ inner product of (1.7) and (1.8) with $(-w_{zz}, -\psi_{zz})^{tr}$. After conducting integration by parts as well as using (1.12) and (1.15), one has

$$\frac{1}{2}\frac{d}{dt}\left(\|w_z\|_2^2 + \|\mathbf{u}_z\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w_z\|_2^2 + \|\omega_z\|_2^2\right) + \epsilon^2\|\psi_{zz}\|_2^2$$
$$\le \int_\Omega |\mathbf{u}_z \cdot \nabla_h w w_z| dxdydz + \int_\Omega |\mathbf{u}\cdot\nabla_h\psi_z\omega_z| dxdydz. \tag{2.8}$$

By Lemma 2.1 with $f = \nabla_h w$, $g = \mathbf{u}_z$, and $h = w_z$, we obtain

$$\int_\Omega |\mathbf{u}_z\cdot\nabla_h w w_z| dxdydz$$
$$\le C\|\Delta_h w\|_2^{1/2}(\|\nabla_h w\|_2 + \|\nabla_h w_z\|_2)^{1/2}\|\mathbf{u}_z\|_2^{1/2}\|\omega_z\|_2^{1/2}\|w_z\|_2$$
$$\le C\left(\|\Delta_h w\|_2 + \|\Delta_h w\|_2^{1/2}\|\nabla_h w_z\|_2^{1/2}\right)\|\mathbf{u}_z\|_2^{1/2}\|\omega_z\|_2^{1/2}\|w_z\|_2$$
$$\le \frac{1}{4Re}\left(\|\nabla_h w_z\|_2^2 + \|\omega_z\|_2^2\right) + C\left(\|\Delta_h w\|_2^2 + \|\mathbf{u}_z\|_2^2\right)\|w_z\|_2^2 + C\|\mathbf{u}_z\|_2^2. \tag{2.9}$$

Also, using Lemma 2.1 with $f = \mathbf{u}$, $g = \nabla_h\psi_z$, and $h = \omega_z$, one has

$$\int_\Omega |\mathbf{u}\cdot\nabla_h\psi_z\omega_z| dxdydz$$
$$\le C\|\omega\|_2^{1/2}(\|\mathbf{u}\|_2 + \|\mathbf{u}_z\|_2)^{1/2}\|\mathbf{u}_z\|_2^{1/2}\|\omega_z\|_2^{3/2}$$
$$\le \frac{1}{4Re}\|\omega_z\|_2^2 + C\|\omega\|_2^2\left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}_z\|_2^2\right)\|\mathbf{u}_z\|_2^2. \tag{2.10}$$

Applying estimates (2.9) and (2.10) to the inequality (2.8) yields

$$\frac{d}{dt}\left(\|w_z\|_2^2 + \|\mathbf{u}_z\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w_z\|_2^2 + \|\omega_z\|_2^2\right) + \epsilon^2\|\psi_{zz}\|_2^2$$
$$\le C\left(\|\Delta_h w\|_2^2 + \|\mathbf{u}_z\|_2^2\right)\|w_z\|_2^2 + C\left(\|\omega\|_2^2\|\mathbf{u}\|_2^2 + \|\omega\|_2^2\|\mathbf{u}_z\|_2^2 + 1\right)\|\mathbf{u}_z\|_2^2.$$

Thanks to Gronwall's inequality, we obtain

$$\|w_z(t)\|_2^2 + \|\mathbf{u}_z(t)\|_2^2 + \int_0^t\left[\frac{1}{Re}\left(\|\nabla_h w_z\|_2^2 + \|\omega_z\|_2^2\right) + \epsilon^2\|\psi_{zz}\|_2^2\right]ds$$
$$\le \left(\|\partial_z w_0\|_2^2 + \|\partial_z\mathbf{u}_0\|_2^2 + C\int_0^t (\|\omega\|_2^2\|\mathbf{u}\|_2^2 + 1)\|\mathbf{u}_z\|_2^2 ds\right) e^{\int_0^t C\left(\|\Delta_h w\|_2^2 + \|\mathbf{u}_z\|_2^2 + \|\omega\|_2^2\|\mathbf{u}_z\|_2^2\right)ds}$$
$$\le C(\|w_0\|_{H^1}, \|\mathbf{u}_0\|_{H^1}). \tag{2.11}$$

The uniform bound (2.11) is due to (2.2), (2.6), and (2.7).

III. RIGOROUS JUSTIFICATION OF THE *A PRIORI* ESTIMATES AND THE EXISTENCE OF STRONG SOLUTIONS

This section is devoted to proving the existence of global strong solutions for model (1.7)–(1.9) by assuming the initial data $(\mathbf{u}_0, w_0)^{tr} \in (H^1(\Omega))^3$ such that $\nabla_h \cdot \mathbf{u}_0 = 0$, $\int_{[0,L]^2} \mathbf{u}_0 dxdy = 0$, and $\int_{[0,L]^2} w_0 dxdy = 0$. We employ the standard Galerkin method and use the analog of the *a priori* estimates that were established in Sec. II.

Let $e_{\mathbf{j}} = \exp(2\pi i[(j_1 x + j_2 y)/L + j_3 z])$ for $\mathbf{j} = (j_1, j_2, j_3)^{tr}$. For $m \in \mathbb{N}$, let $P_m\left(L^2(\Omega)\right)$ be a subspace of $L^2(\Omega)$ spanned by $\{e_{\mathbf{j}}\}_{|\mathbf{j}|\le m}$. Also, for any $L^2(\Omega)$ function $f = \sum_{\mathbf{j}\in\mathbb{Z}^3} \alpha_{\mathbf{j}} e_{\mathbf{j}}$, with $\alpha_{\mathbf{j}} = (f, e_{\mathbf{j}})$, we write $P_m f = \sum_{|\mathbf{j}|\le m} \alpha_{\mathbf{j}} e_{\mathbf{j}}$.

Let us consider the Galerkin approximation for our model (1.7)–(1.9),

$$\partial_t w_m + P_m(\mathbf{u}_m \cdot \nabla_h w_m) - \partial_z \psi_m = \frac{1}{Re}\Delta_h w_m, \tag{3.1}$$

$$\partial_t \omega_m + P_m(\mathbf{u}_m \cdot \nabla_h \omega_m) - \partial_z w_m = \frac{1}{Re}\Delta_h \omega_m + \epsilon^2 \partial_{zz}\psi_m, \tag{3.2}$$

$$\nabla_h \cdot \mathbf{u}_m = 0, \tag{3.3}$$

$$\mathbf{u}_m(0) = P_m \mathbf{u}_0, \quad w_m(0) = P_m w_0, \tag{3.4}$$

where $\mathbf{u}_m, w_m \in P_m\left(L^2(\Omega)\right)$ and $\omega_m = \nabla_h \times \mathbf{u}_m$ and $\psi_m = (-\Delta_h)^{-1}\omega_m$ with $\int_{[0,L]^2} \psi_m dxdy = 0$.

For each $m \ge 1$, Galerkin approximation (3.1)–(3.4) corresponds to a first order system of ordinary differential equations with quadratic nonlinearity. Therefore, by the theory of ordinary differential equations, there exists some $T_m > 0$ such that system (3.1)–(3.4) admits a unique solution $(\mathbf{u}_m, w_m)^{tr}$ on $[0, T_m]$. Since $\mathbf{u}_m$ and w_m have finitely many modes, they are smooth functions, and therefore all of the *a priori* estimates established in Sec. II are valid for the Galerkin approximate solution $(\mathbf{u}_m, w_m)^{tr}$. In particular, the $H^1(\Omega)$ norm of $(\mathbf{u}_m, w_m)^{tr}$ is uniformly bounded for all time. Hence, the Galerkin approximate solution $(\mathbf{u}_m, w_m)^{tr}$ exists globally in time, in particular, over $[0, T]$, for every $T > 0$.

Furthermore, by the *a priori* estimates in Sec. II, one has the following uniform bounds for the sequence of the Galerkin approximate solutions:

$$\mathbf{u}_m,\ w_m \text{ are uniformly bounded in } L^\infty(0, T; H^1(\Omega)), \tag{3.5}$$

$$\nabla_h \omega_m,\ \Delta_h w_m,\ \partial_z \omega_m,\ \nabla_h \partial_z w_m,\ \partial_{zz}\psi_m \text{ are uniformly bounded in } L^2(\Omega \times (0, T)). \tag{3.6}$$

Therefore, there exists a subsequence, denoted also by $\mathbf{u}_m, w_m, \omega_m, \psi_m$, and corresponding limits, $\mathbf{u}$, w, ω, and ψ, respectively, such that

$$\mathbf{u}_m \to \mathbf{u},\ \ w_m \to w,\ \text{ weakly}^* \text{in } L^\infty(0, T; H^1(\Omega)), \tag{3.7}$$

$$\nabla_h \omega_m \to \nabla_h \omega,\ \ \Delta_h w_m \to \Delta_h w,\ \text{ weakly in } L^2(\Omega \times (0, T)), \tag{3.8}$$

$$\partial_z \omega_m \to \partial_z \omega,\ \ \nabla_h \partial_z w_m \to \nabla_h w_z,\ \ \partial_{zz}\psi_m \to \partial_{zz}\psi,\ \text{ weakly in } L^2(\Omega \times (0, T)). \tag{3.9}$$

Moreover, due to the *a priori* estimates in Sec. II, we find that

$$\sup_{0\le t\le T}\left(\|\mathbf{u}_m(t)\|^2_{H^1(\Omega)} + \|w_m(t)\|^2_{H^1(\Omega)}\right) \le K, \tag{3.10}$$

where K is independent of T but depends only on parameters Re, ϵ, L and the H^1-norms $\|\mathbf{u}_0\|_{H^1(\Omega)}$ and $\|w_0\|_{H^1(\Omega)}$ of the initial data. Also thanks to the weak-$*$ convergence stated in (3.7), one has $\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))} \le \liminf_{m\to\infty} \|\mathbf{u}_m\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|w\|_{L^\infty(0,T;H^1(\Omega))} \le \liminf_{m\to\infty} \|w_m\|_{L^\infty(0,T;H^1(\Omega))}$. Therefore, we obtain from (3.10) that

$$\sup_{0\le t\le T}\left(\|\mathbf{u}(t)\|^2_{H^1(\Omega)} + \|w(t)\|^2_{H^1(\Omega)}\right) \le K.$$

In order to obtain the strong convergence of the approximate solutions, we shall derive uniform bounds for $\partial_t w_m$ and $\partial_t \mathbf{u}_m$. First, we claim that the sequence $\partial_t w_m$ is uniformly bounded in $L^2(\Omega \times (0,T))$. Indeed, for any function $\varphi \in L^{4/3}(0,T;L^2(\Omega))$, we use Lemma 2.1 to estimate

$$\begin{aligned}
&\int_0^T \int_\Omega |(\mathbf{u}_m \cdot \nabla_h w_m)\varphi| dxdydzdt \\
&\leq C \int_0^T \|\omega_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h w_m\|_2^{1/2} \|\Delta_h w_m\|_2^{1/2} \|\varphi\|_2 dt \\
&\leq C \sup_{t\in[0,T]} \left(\|\omega_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h w_m\|_2^{1/2} \right) \\
&\quad \cdot \left(\int_0^T \|\Delta_h w_m\|_2^2 dt \right)^{1/4} \left(\int_0^T \|\varphi\|_2^{4/3} dt \right)^{3/4} \\
&\leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}) \|\varphi\|_{L^{4/3}(0,T;L^2(\Omega))},
\end{aligned} \tag{3.11}$$

where the last inequality is due to the *a priori* estimates (2.2), (2.6), (2.7), and (2.11). Consequently, the sequence

$$\mathbf{u}_m \cdot \nabla_h w_m \text{ is uniformly bounded in } L^4(0,T;L^2(\Omega)). \tag{3.12}$$

As a result, from (3.5), (3.6), and (3.12), we obtain from (3.1) that the sequence

$$\partial_t w_m \text{ is uniformly bounded in } L^2(\Omega \times (0,T)). \tag{3.13}$$

Next, we show that $\partial_t \mathbf{u}_m$ is uniformly bounded in $L^2(\Omega \times (0,T))$. Recall the Hilbert space $H_h^1(\Omega) = \{f \in L^2(\Omega) : \nabla_h f \in L^2(\Omega)\}$ associated with the norm $\|f\|_{H_h^1(\Omega)}^2 = \|f\|_2^2 + \|\nabla_h f\|_2^2$. For any function $\phi \in L^2(0,T;H_h^1(\Omega))$, we apply Lemma 2.1 in order to estimate

$$\begin{aligned}
&\int_0^T \int_\Omega |(\mathbf{u}_m \cdot \nabla_h \omega_m)\phi| dxdydzdt \\
&\leq C \int_0^T \|\omega_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \|\nabla_h \omega_m\|_2 \|\phi\|_2^{1/2} (\|\phi\|_2 + \|\nabla_h \phi\|_2)^{1/2} dt \\
&\leq C \sup_{t\in[0,T]} \left(\|\omega_m\|_2^{1/2} (\|\mathbf{u}_m\|_2 + \|\partial_z \mathbf{u}_m\|_2)^{1/2} \right) \\
&\quad \cdot \left(\int_0^T \|\nabla_h \omega_m\|_2^2 dt \right)^{1/2} \left(\int_0^T \left(\|\phi\|_2^2 + \|\nabla_h \phi\|_2^2 \right) dt \right)^{1/2} \\
&\leq C(\|\mathbf{u}_0\|_{H^1}, \|w_0\|_{H^1}) \|\phi\|_{L^2(0,T;H_h^1(\Omega))},
\end{aligned} \tag{3.14}$$

where we have used the *a priori* estimates (2.2), (2.6), and (2.11). Therefore, the sequence

$$\mathbf{u}_m \cdot \nabla_h \omega_m \text{ is uniformly bounded in } L^2(0,T;H_h^1(\Omega)'), \tag{3.15}$$

where $(H_h^1(\Omega))'$ is the dual space of $H_h^1(\Omega)$. Consequently, according to (3.5), (3.6), and (3.15), we obtain from (3.2) that the sequence

$$\partial_t \omega_m \text{ is uniformly bounded in } L^2(0,T;H_h^1(\Omega)'), \tag{3.16}$$

and thus

$$\partial_t \mathbf{u}_m \text{ is uniformly bounded in } L^2(\Omega \times (0,T)). \tag{3.17}$$

Then, we infer from (3.13) and (3.17) that there is a subsequence such that

$$\partial_t w_m \to \partial_t w, \;\; \partial_t \mathbf{u}_m \to \partial_t \mathbf{u} \;\text{ weakly in } L^2(\Omega \times (0,T)). \tag{3.18}$$

By (3.5), (3.13), and (3.17), and thanks to Aubin's compactness theorem, we have, for a subsequence, that the following strong convergence holds,

$$\mathbf{u}_m \to \mathbf{u}, \;\; w_m \to w \;\text{ in } L^2(\Omega \times (0,T)). \tag{3.19}$$

Next, we show the convergence of the nonlinear terms in (3.1) and (3.2). Let η be a trigonometric polynomial with continuous coefficients. For m larger than the degree of η, we have

$$\begin{aligned} &\int_0^T \int_\Omega P_m(\mathbf{u}_m \cdot \nabla_h \omega_m)\eta dxdydzdt \\ &= \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla_h \omega_m)\eta dxdydzdt + \int_0^T \int_\Omega ((\mathbf{u}_m - \mathbf{u}) \cdot \nabla_h \omega_m)\eta dxdydzdt. \end{aligned} \tag{3.20}$$

Since $\nabla_h \omega_m \to \nabla_h \omega$ weakly in $L^2(\Omega \times (0, T))$, $\mathbf{u}_m \to \mathbf{u}$ in $L^2(\Omega \times (0, T))$, and $\nabla_h \omega_m$ is uniformly bounded in $L^2(\Omega \times (0, T))$, we can pass to the limit in (3.20),

$$\lim_{m\to\infty} \int_0^T \int_\Omega P_m(\mathbf{u}_m \cdot \nabla_h \omega_m)\eta dxdydzdt = \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla_h \omega)\eta dxdydzdt. \tag{3.21}$$

An analogous argument yields

$$\lim_{m\to\infty} \int_0^T \int_\Omega P_m(\mathbf{u}_m \cdot \nabla_h w_m)\eta dxdydzdt = \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla_h w)\eta dxdydzdt. \tag{3.22}$$

Therefore, due to (3.7)–(3.9), (3.18), (3.21), and (3.22), we pass to the limit for the Galerkin approximate Eqs. (3.1)–(3.3). It follows that

$$\int_0^T \int_\Omega \left(\partial_t w + \mathbf{u} \cdot \nabla_h w - \partial_z \psi - \frac{1}{Re}\Delta_h w\right)\eta dxdydzdt = 0, \tag{3.23}$$

$$\int_0^T \int_\Omega \left(\partial_t \omega + \mathbf{u} \cdot \nabla_h \omega - \partial_z w - \frac{1}{Re}\Delta_h \omega - \epsilon^2 \partial_{zz}\psi\right)\eta dxdydzdt = 0, \tag{3.24}$$

with $\nabla_h \cdot \mathbf{u} = 0$, for any trigonometric polynomial η with continuous coefficients.

By using an estimate similar to (3.11), we can deduce $\mathbf{u} \cdot \nabla_h w \in L^4(0, T; L^2(\Omega))$. Also, using an estimate similar to (3.14), one may derive $\mathbf{u} \cdot \nabla_h \omega \in L^2(0,T;H_h^1(\Omega)')$. Then we conclude from (3.23) and (3.24) that (1.17) and (1.18) hold.

It remains to verify the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $w(0) = w_0$. Indeed, one infers from (3.19) that, on a subsequence, $w_m(t) \to w(t)$ in $L^2(\Omega)$ for a.e. $t \in [0, T]$. We multiply (3.1) by a trigonometric polynomial ξ and integrate over the region $\Omega \times [0, t]$ and then pass to the limit using (3.4), (3.7), (3.8), and (3.22). It follows that

$$(w(t) - w_0, \xi) + \int_0^t \int_\Omega \left(\mathbf{u} \cdot \nabla_h w - \partial_z \psi - \frac{1}{Re}\Delta_h w\right)\xi dxdydzds = 0, \tag{3.25}$$

for a.e. $t \in [0, T]$. Now, we multiply (1.17) with ξ and integrate over $\Omega \times [0, t]$. We then compare the resulting equality with (3.25) to get $(w_0, \xi) = (w(0), \xi)$ for any trigonometric polynomial ξ, which implies $w(0) = w_0$. Similarly, one can verify $\mathbf{u}(0) = \mathbf{u}_0$.

Finally, due to the regularity of solutions, we can multiply (1.17) and (1.18) by $(w,\psi)^{tr}$ and integrate the result over $\Omega \times (0, t)$ for $t \in [0, T]$. Then the energy identity (1.19) follows.

IV. UNIQUENESS OF STRONG SOLUTIONS

This section is devoted to proving that strong solutions for the system (1.7)–(1.9) are unique and depend continuously on the initial data. Assume that there are two strong solutions $(\mathbf{u}_1, w_1)^{tr}$ and $(\mathbf{u}_2, w_2)^{tr}$ on $[0, T]$ in the sense of Definition 1.1. Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $w = w_1 - w_2$. Therefore,

$$\frac{\partial w}{\partial t} + \mathbf{u}\cdot\nabla_h w_1 + \mathbf{u}_2\cdot\nabla_h w - \frac{\partial \psi}{\partial z} = \frac{1}{Re}\Delta_h w, \ \text{in } L^2(\Omega\times(0,T)), \tag{4.1}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla_h \omega_1 + \mathbf{u}_2\cdot\nabla_h \omega - \frac{\partial w}{\partial z} = \frac{1}{Re}\Delta_h \omega + \epsilon^2\frac{\partial^2\psi}{\partial z^2}, \ \text{in } L^2(0,T;H_h^1(\Omega)'), \tag{4.2}$$

with $\nabla_h\cdot\mathbf{u} = 0$.

Since $\mathbf{u}$ and w satisfy the regularity (1.16), we can multiply (4.1) and (4.2) by $(w,\psi)^{tr}$ and integrate over Ω. By using (1.11), (1.12), (1.14), and (1.15), we obtain, for a.e. $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|w\|_2^2 + \|\mathbf{u}\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + \epsilon^2\|\psi_z\|_2^2\\ &\le \int_\Omega |(\mathbf{u}\cdot\nabla_h w)w_1|dxdydz + \int_\Omega |(\mathbf{u}_2\cdot\nabla_h\psi)\omega|dxdydz.\end{aligned} \tag{4.3}$$

Next we estimate the two integrals on the right-hand side of (4.3).

Using Lemma 2.1 with $f = w_1$, $g = \mathbf{u}$, and $h = \nabla_h w$, we obtain

$$\begin{aligned}&\int_\Omega |(\mathbf{u}\cdot\nabla_h w)w_1|dxdydz\\ &\le C\|\nabla_h w_1\|_2^{1/2}\left(\|w_1\|_2 + \|\partial_z w_1\|_2\right)^{1/2}\|\mathbf{u}\|_2^{1/2}\|\nabla_h\mathbf{u}\|_2^{1/2}\|\nabla_h w\|_2\\ &\le \frac{1}{4Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + C\|\nabla_h w_1\|_2^2\left(\|w_1\|_2^2 + \|\partial_z w_1\|_2^2\right)\|\mathbf{u}\|_2^2.\end{aligned} \tag{4.4}$$

Also, using Lemma 2.1 with $f = \mathbf{u}_2$, $g = \nabla_h\psi$, $h = \omega$, we have

$$\begin{aligned}\int_\Omega |(\mathbf{u}_2\cdot\nabla_h\psi)\omega|dxdydz &\le C\|\omega_2\|_2^{1/2}(\|\mathbf{u}_2\|_2 + \|\partial_z\mathbf{u}_2\|_2)^{1/2}\|\mathbf{u}\|_2^{1/2}\|\nabla_h\mathbf{u}\|_2^{3/2}\\ &\le \frac{1}{4Re}\|\nabla_h\mathbf{u}\|_2^2 + C\|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2 + \|\partial_z\mathbf{u}_2\|_2^2)\|\mathbf{u}\|_2^2.\end{aligned} \tag{4.5}$$

Now, we combine the estimates (4.3)–(4.5) to deduce, for a.e. $t\in[0,T]$,

$$\begin{aligned}&\frac{d}{dt}\left(\|w\|_2^2 + \|\mathbf{u}\|_2^2\right) + \frac{1}{Re}\left(\|\nabla_h w\|_2^2 + \|\nabla_h\mathbf{u}\|_2^2\right) + \epsilon^2\|\psi_z\|_2^2\\ &\le C\left[\|\nabla_h w_1\|_2^2\left(\|w_1\|_2^2 + \|\partial_z w_1\|_2^2\right) + \|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2 + \|\partial_z\mathbf{u}_2\|_2^2)\right]\|\mathbf{u}\|_2^2.\end{aligned}$$

By Gronwall's inequality, it follows that

$$\begin{aligned}&\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2\\ &\le \left(\|w(0)\|_2^2 + \|\mathbf{u}(0)\|_2^2\right)e^{C\int_0^t \|\nabla_h w_1\|_2^2(\|w_1\|_2^2+\|\partial_z w_1\|_2^2)+\|\omega_2\|_2^2(\|\mathbf{u}_2\|_2^2+\|\partial_z\mathbf{u}_2\|_2^2)ds}\\ &\le \left(\|w(0)\|_2^2 + \|\mathbf{u}(0)\|_2^2\right)e^{tC\left(\|w_1(0)\|_{H^1},\|\mathbf{u}_2(0)\|_{H^1}\right)},\end{aligned} \tag{4.6}$$

for any $t\in[0,T]$. In particular, if $(\mathbf{u}(0), w(0))^{tr} = 0$, i.e., the initial values of the two solutions $(\mathbf{u}_1, w_1)^{tr}$ and $(\mathbf{u}_2, w_2)^{tr}$ coincide, then (4.6) implies $\|w(t)\|_2^2 + \|\mathbf{u}(t)\|_2^2 = 0$ for all $t\in[0,T]$. This completes the proof for the uniqueness of strong solutions.

To see the continuous dependence on the initial data, we let $(\tilde{\mathbf{u}}_0, \tilde{w}_0)^{tr}\in(H^1(\Omega))^3$ and take a bounded sequence $\{(\mathbf{u}_0^n, w_0^n)^{tr}\}$ of initial data in $H^1(\Omega)$ such that $(\mathbf{u}_0^n, w_0^n)^{tr}\to(\tilde{\mathbf{u}}_0,\tilde{w}_0)^{tr}$ in $L^2(\Omega)$ and $\|\mathbf{u}_0^n\|_{H^1}$, $\|w_0^n\|_{H^1}$, $\|\tilde{\mathbf{u}}_0\|_{H^1}$, $\|\tilde{w}_0\|_{H^1}\le M$ for some $M > 0$. Denote the corresponding strong solutions by $(\mathbf{u}^n, w^n)^{tr}$ and $(\tilde{\mathbf{u}}, \tilde{w})^{tr}$, respectively. Then, on account of (4.6), we have, for all $t\in[0,T]$,

$$\begin{aligned}\|\tilde{w} - w^n\|_2^2 + \|\tilde{\mathbf{u}} - \mathbf{u}^n\|_2^2 &\le \left(\|\tilde{w}_0 - w_0^n\|_2^2 + \|\tilde{\mathbf{u}}_0 - \mathbf{u}_0^n\|_2^2\right)e^{tC\left(\|\tilde{w}_0\|_{H^1},\|\mathbf{u}_0^n\|_{H^1}\right)}\\ &\le \left(\|\tilde{w}_0 - w_0^n\|_2^2 + \|\tilde{\mathbf{u}}_0 - \mathbf{u}_0^n\|_2^2\right)e^{T\cdot C(M)}.\end{aligned}$$

It follows that $(\mathbf{u}^n, w^n)^{tr} \to (\tilde{\mathbf{u}}, \tilde{w})^{tr}$ in $C([0, T]; L^2(\Omega))$. This completes the proof for the continuous dependence on the initial data with respect to the L^2-norm for the strong solutions.

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APPENDIX: PROOF OF LEMMA 2.1

We prove the anisotropic Ladyzhenskaya type inequality stated in Lemma 2.1.

Proof. It suffices to prove the inequality in Lemma 2.1 for smooth periodic functions and then pass to the limit using a standard density argument. Recall $\Omega = [0, L]^2 \times [0, 1]$. For a fixed $(x, y) \in [0, L]^2$ and for every $z, \sigma \in [0, 1]$, we have

$$\begin{aligned} f^4(x,y,z) &= \int_\sigma^z \frac{d}{d\xi}\Big(f^4(x,y,\xi)\Big)d\xi + f^4(x,y,\sigma) \\ &\le 4\int_0^1 |f(x,y,\xi)|^3|f_\xi(x,y,\xi)|d\xi + f^4(x,y,\sigma). \end{aligned} \tag{A1}$$

Integrating (A1) with respect to σ over $[0, 1]$, we obtain

$$f^4(x,y,z) \le 4\int_0^1 |f(x,y,\xi)|^3|f_\xi(x,y,\xi)|d\xi + \int_0^1 f^4(x,y,\sigma)d\sigma,$$

and by the Cauchy-Schwarz inequality, we have

$$f^4(x,y,z) \le 4\|f\|_{L_z^6}^3\|f_z\|_{L_z^2} + \|f\|_{L_z^4}^4, \tag{A2}$$

where $\|f\|_{L_z^p} = \left(\int_0^1 |f(x,y,z)|^p dz\right)^{1/p}$ if $p > 1$. Also we denote $\|f\|_{L_z^\infty} = \sup_{z\in[0,1]} |f(x,y,z)|$. Then, inequality (A2) can be written as

$$\|f\|_{L_z^\infty} \le C\|f\|_{L_z^6}^{3/4}\|f_z\|_{L_z^2}^{1/4} + \|f\|_{L_z^4}. \tag{A3}$$

Thanks to Hölder's inequality and (A3), we have

$$\begin{aligned} \int_\Omega |fgh|dxdydz &\le \int_{[0,L]^2} \|f\|_{L_z^\infty}\|g\|_{L_z^2}\|h\|_{L_z^2}dxdy \\ &\le C\int_{[0,L]^2}\Big(\|f\|_{L_z^6}^{3/4}\|f_z\|_{L_z^2}^{1/4} + \|f\|_{L_z^4}\Big)\|g\|_{L_z^2}\|h\|_{L_z^2}dxdy \\ &\le C\Big(\|f\|_6^{3/4}\|f_z\|_2^{1/4} + \|f\|_4\Big)\left(\int_{[0,L]^2}\|g\|_{L_z^2}^4 dxdy\right)^{1/4}\|h\|_2. \end{aligned} \tag{A4}$$

Recall the Ladyzhenskaya inequality (see, e.g., Ref. 15) in the three-dimensional periodic domain Ω,

$$\|\varphi\|_p \le C_p\|\varphi\|_2^{\frac{6-p}{2p}}\left(\|\varphi\|_2 + \|\varphi_x\|_2\right)^{\frac{p-2}{2p}}\left(\|\varphi\|_2 + \|\varphi_y\|_2\right)^{\frac{p-2}{2p}}\left(\|\varphi\|_2 + \|\varphi_z\|_2\right)^{\frac{p-2}{2p}}, \tag{A5}$$

for $\varphi \in H^1(\Omega)$ and $p \in [2,6]$. By (A5), one has

$$\|f\|_6 \le C(\|f\|_2 + \|\nabla_h f\|_2)^{2/3}(\|f\|_2 + \|f_z\|_2)^{1/3} \tag{A6}$$

and

$$\|f\|_4 \le C\|f\|_2^{1/4}(\|f\|_2 + \|\nabla_h f\|_2)^{1/2}(\|f\|_2 + \|f_z\|_2)^{1/4}. \tag{A7}$$

By virtue of (A6) and (A7), we have

$$\|f\|_6^{3/4}\|f_z\|_2^{1/4} + \|f\|_4 \le C\left(\|f\|_2 + \|\nabla_h f\|_2\right)^{1/2}\left(\|f\|_2 + \|f_z\|_2\right)^{1/2}. \tag{A8}$$

Recall Agmon's inequality (see, e.g., Ref. 1) in one dimension,

$$\|\phi\|_{L^\infty([0,L])} \le C\|\phi\|_{L^2([0,L])}^{1/2}\|\phi\|_{H^1([0,L])}^{1/2}. \tag{A9}$$

By using (A9), we obtain

$$\begin{aligned}
\int_{[0,L]^2}\|g\|_{L_z^2}^4 dxdy &\le \left(\int_0^L\int_0^1 \sup_{x\in[0,L]} g^2 dzdy\right)\left(\int_0^L\int_0^1 \sup_{y\in[0,L]} g^2 dzdx\right)\\
&\le C\left[\int_0^L\int_0^1\left(\int_0^L g^2dx\right)^{\frac{1}{2}}\left(\int_0^L\left(g^2+g_x^2\right)dx\right)^{\frac{1}{2}}dzdy\right]\\
&\quad\cdot\left[\int_0^L\int_0^1\left(\int_0^L g^2dy\right)^{\frac{1}{2}}\left(\int_0^L\left(g^2+g_y^2\right)dy\right)^{\frac{1}{2}}dzdx\right]\\
&\le C\|g\|_2^2\left(\|g\|_2^2+\|\nabla_h g\|_2^2\right).
\end{aligned} \tag{A10}$$

By combining (A4), (A8), and (A10), we conclude the proof of Lemma 2.1. □

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