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Inertial Manifolds for Certain Subgrid-Scale α-Models of Turbulence*

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Abstract. In this paper we prove the existence of an inertial manifold, i.e., a globally invariant, exponentially attracting, finite-dimensional smooth manifold, for two different subgrid-scale α-models of turbulence, the simplified Bardina model and the modified Leray-α model, in two-dimensional space. That is, we show the existence of an exact rule that parameterizes the dynamics of small spatial scales in terms of the dynamics of large ones. In particular, this implies that the long-time dynamics of these turbulence models is equivalent to that of a finite-dimensional system of ordinary differential equations.

Key words. inertial manifold, turbulence models, subgrid-scale models, Navier–Stokes equations, modified Leray-α model, simplified Bardina model

AMS subject classifications. 35Q30, 37L30, 76B03, 76F20, 76F55, 76F65

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1. Introduction.

The fidelity of the Navier–Stokes equations (NSE) is in capturing the dynamics of turbulent flow. However, their downfall is in reliable direct numerical simulation of turbulence. Therefore scientists have developed various approximate models which are computable and preserve some statistical properties of the physical phenomenon of turbulence, and of particular interest to us in this paper are certain subgrid-scale α-models of turbulence.

In many applications, it is enough to capture the mean features of the flow; to obtain this we need to average the nonlinear term in the NSE, which leads to the well-known closure problem. In 1980, Bardina, Ferziger, and Reynolds [3] introduced a particular subgrid-scale model, later simplified by Layton and Lewandowski (see [30]), which takes the form

$$\begin{cases} v_t - \nu\Delta v + (\bar{v}\cdot\nabla)\bar{v} + \nabla p = f, \\ \nabla\cdot v = 0, \\ v = \bar{v} - \alpha^2\Delta\bar{v}. \end{cases} \tag{1}$$

Here the unknowns are the fluid velocity field v and the "filtered" velocity vector $\bar{v}$, as well as the "filtered" pressure scalar p. In addition, there are two given parameters: $\nu > 0$ is the

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constant kinematic viscosity, and $\alpha > 0$ is the length scale parameter which represents the width of the filter. The vector field f is a given body forcing, assumed to be time independent. For more details about model (1), see [4, 6, 28, 29].

In 2005, Cheskidov et al. [10] introduced the Leray-α model:

$$\begin{cases} w_t - \nu\Delta w + (\bar{w}\cdot\nabla)w + \nabla p = f, \\ \nabla\cdot w = 0, \\ w = \bar{w} - \alpha^2\Delta\bar{w}. \end{cases} \tag{2}$$

Leray in 1934 [31] established the well-posedness of the NSE in two and three dimensions, by introducing a modified system similar to (2), for which it was easier to prove the existence and uniqueness of solutions; then, by passing with the parameter $\alpha \to 0^+$, he achieved the existence of solutions to the NSE. An upper bound of the dimension of the global attractor and an analysis of the energy spectrum of the solutions of the three-dimensional (3D) version of (2) were established in [10], which suggested that the Leray-α model has great potential to become a good subgrid-scale large-eddy simulation model of turbulence. See also a computational study of this model in [23, 33, 34].

Inspired by the remarkable performance of the Leray-α model, Ilyin, Lunasin, and Titi [24] proposed a modified-Leray-α model:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)\bar{u} + \nabla p = f, \\ \nabla\cdot u = 0, \\ u = \bar{u} - \alpha^2\Delta\bar{u}. \end{cases} \tag{3}$$

It was demonstrated in [24] that the reduced modified-Leray-α model (3) in infinite channels and pipes is equally impressive as a closure model for Reynolds averaged equations as the Leray-α model (2) and other subgrid-scale α-models, e.g., the Navier–Stokes-α (also known as the viscous Camassa–Holm equations [7, 8, 9, 15, 16]) and the Clark-α [5].

Comparing the three turbulence models (1), (2), and (3), we see that in the simplified Bardina model (1), both arguments of the nonlinearity are regularized, while the Leray-α model (2) regularizes only the first argument of the nonlinear term, i.e., the transport velocity, and in the modified Leray-α model (3), solely the second argument of the nonlinearity is smoothed; i.e., the transported velocity is regularized. For the models (1), (2), and (3), the global well-posedness in three dimensions, the existence of a finite-dimensional global attractor, and the analysis of their energy spectra have been established in [6, 10, 24, 31].

Our interest lies in the large-time behavior of the dynamics generated by turbulence models. In particular, we aim to show existence of inertial manifolds for two different systems in two dimensions: the simplified Bardina model (1) and the modified-Leray-α model (3), subject to periodic boundary condition, with basic domain $\Omega = [0, 2\pi L]^2$.

The long-time behavior of solutions of a large class of dissipative PDEs possesses a resemblance to the behavior of finite-dimensional systems. The concept of an inertial manifold was introduced to capture such phenomena. Indeed, an inertial manifold of an evolution equation is a finite-dimensional Lipschitz invariant manifold attracting *exponentially* all the trajectories of a dynamical system induced by the underlying evolution equation [20, 21]. The precise definition is given in section 3.2. The existence of an inertial manifold for an infinite-dimensional evolution equation represents the best analytical form of reduction of an infinite system to a finite-dimensional one. This is because an inertial manifold is finite-dimensional, and the restriction of the evolutionary equation to this manifold reduces to a finite system of ODEs, which is called the *inertial form* of the given evolutionary equation. As a result, the dynamical properties of the solution of the evolutionary PDE, which is an infinite-dimensional dynamical system, can be analyzed by the study of an inertial form, which is a finite-dimensional system.

Inertial manifolds were introduced by Foias, Sell, and Temam in [20, 21]. The idea was employed on a large class of dissipative equations [22] (see also [40]). A number of dynamical systems possess inertial manifolds, such as certain nonlinear reaction-diffusion equations in two [14, 21, 36] and in three [35] dimensions, the Kuramoto–Sivashinsky equation [19, 20, 22, 40], the Cahn–Hilliard equation [13], and the von Kármán plate equations [11]. One may refer to [13] for the study of inertial manifolds for many dissipative PDEs. It is worth mentioning that an original motivation for the theory of inertial manifolds was treating the NSE. Unfortunately, the problem of existence of inertial manifolds for the two-dimensional (2D) NSE is still unsolved, and we are unaware of any such result for a system of hydrodynamics which does not involve an artificial hyperviscosity. In particular, the question of existence of an inertial manifold is still open even for the 2D Navier–Stokes-α, Leray-α, and Clark-α models. Recently, the concept of *determining form* was introduced in [17, 18], in which it was shown that the long-time dynamics of such models, especially that of the 2D NSE, is equivalent to the long-time dynamics of an ODE with continuously Lipschitz vector field in certain infinite-dimensional space of trajectories with finite range (see also [25] for related results). In this paper, we succeed in obtaining the existence of inertial manifolds for the simplified Bardina model (1) and the modified Leray-α model (3), since the nonlinear terms in these two systems are milder than those of the NSE and other α-models of turbulence.

The paper is organized as follows: section 2 is devoted to the preliminaries and the functional settings. In sections 3 and 4, we study the simplified Bardina model (1) and the modified Leray-α model (3), respectively, and prove the existence of absorbing balls in various Hilbert spaces, as well as the existence of an inertial manifold for both models. In the appendix, we give a detailed justification of the strong squeezing property for these two systems.

2. Preliminaries.

We introduce some preliminary background material, which is standard in the mathematical theory of the NSE.

(i) Let $\mathcal{F}$ be the set of all 2D trigonometric vector-valued polynomials with periodic domain Ω. We then set

$$\mathcal{V} = \left\{ \phi \in \mathcal{F} : \nabla \cdot \phi = 0 \text{ and } \int_\Omega \phi(x)\ dx = 0 \right\}.$$

We set H and V to be the closures of $\mathcal{V}$ in $(L^2_{per}(\Omega))^2$ and $(H^1_{per}(\Omega))^2$, respectively.

(ii) We denote by $P_\sigma : (L^2_{per}(\Omega))^2 \to H$ the Helmholtz–Leray orthogonal projection operator, and by $A = -P_\sigma \Delta$ the Stokes operator with the domain $D(A) = (H^2_{per}(\Omega))^2 \cap V$. Since we work with periodic space, it is known that

$$Au = -P_\sigma \Delta u = -\Delta u \quad \text{ for all } u \in D(A).$$

The operator A^{-1} is a self-adjoint positive definite compact operator from H into H

(cf. [12, 41]). We denote by $0 < L^{-2} = \lambda_1 \le \lambda_2 \le \cdots$ the eigenvalues of A, repeated according to their multiplicities.

(iii) We denote by $|\cdot|$ and $(\cdot,\cdot)$ the L^2_{per} norm and the L^2_{per} inner product, respectively. Moreover, one can show that $V = D(A^{1/2})$. Therefore we denote by $((\cdot,\cdot)) = (A^{1/2}\cdot, A^{1/2}\cdot)$ and $||\cdot|| = |A^{1/2}\cdot|$ the inner product and the norm, respectively, on V. We also observe that $D(A^{s/2}) = (H^s_{per}(\Omega))^2 \cap V$ (cf. [12, 41]). In addition, we denote by V' the dual space of V, and by $D(A)'$ the dual space of $D(A)$.

(iv) For $r < s$, we recall the following version of Poincaré's inequality:

$$\lambda_1^{s-r}|A^r\phi| \le |A^s\phi| \tag{4}$$

for every $\phi \in D(A^s)$.

(v) For $w_1, w_2 \in V$ we define the bilinear form

$$B(w_1, w_2) = P_\sigma((w_1 \cdot \nabla)w_2).$$

The bilinear form $B : V \times V \to V'$ is continuous, and it satisfies

$$\langle B(w_1, w_2), w_3\rangle_{V'} = -\langle B(w_1, w_3), w_2\rangle_{V'}. \tag{5}$$

In particular, $\langle B(w_1, w_2), w_2\rangle_{V'} = 0$. Moreover, $(B(w,w), Aw) = 0$ for every $w \in D(A)$ (this is true only in the 2D periodic case). See [12, 40, 41, 42] for proofs. In addition, we shall use the following estimate on the L^2 norm of $B(w_1, w_2)$ in two dimensions:

$$|B(w_1, w_2)| \le c|w_1|^{\frac{1}{2}}\|w_1\|^{\frac{1}{2}}\|w_2\|^{\frac{1}{2}}|Aw_2|^{\frac{1}{2}}, \tag{6}$$

which is due to Hölder's inequality and Ladyzhenskaya's inequality in two dimensions, $|\phi|_{L^4} \le c|\phi|^{\frac{1}{2}}\|\phi\|^{\frac{1}{2}}$.

Finally, we quote the following classical result (see, e.g., [40, 41]).

Lemma 1. *Let $X \subset H \equiv H' \subset X'$ be Hilbert spaces. If $u \in L^2(0,T;X)$ with $u_t \in L^2(0,T;X')$, then u is almost everywhere equal to an absolutely continuous function from $[0,T]$ into H, and the following equality holds in the distribution sense on $(0,T)$:*

$$\frac{d}{dt}|u|_H^2 = 2\langle u_t, u\rangle_{X'}. \tag{7}$$

3. The simplified Bardina model.

This section is devoted to proving the existence of an inertial manifold for the 2D simplified Bardina model. We apply the Helmholtz–Leray orthogonal projection P_σ to (1) and obtain the following equivalent functional differential equation (see, e.g., [12, 41]):

$$\begin{cases} v_t + \nu Av + B(\bar{v}, \bar{v}) = f, \\ v = \bar{v} + \alpha^2 A\bar{v}, \\ v(0) = v_0. \end{cases} \tag{8}$$

Moreover, we assume that the forcing term and the initial data have spatial zero mean, i.e., $\int_\Omega f(x)dx = \int_\Omega v_0(x)dx = 0$, and hence $\int_\Omega v(x,t)dx = 0$ for all $t \ge 0$.

In [6], Cao, Lunasin, and Titi proved the global well-posedness of the 3D viscous simplified Bardina model (8), as well as the existence of a finite-dimensional global attractor. Therefore we will not discuss here the question of well-posedness and the attractor's dimension, because the 2D case follows similar treatment. Notably, the global regularity of the 3D inviscid simplified Bardina model was also shown in [6], i.e., when $\nu = 0$. In this inviscid case, model (8) coincides with the inviscid Navier–Stokes–Voigt model, namely, the Euler–Voigt model which has been a subject of intensive recent analytical and computational studies (cf. [26, 27, 28, 29, 32, 33, 37]).

Now we can quote the following theorem without proof (since it has been proven in the 3D case in [6]), which states the global existence and uniqueness of regular solutions of (8).

Theorem 2 (regular solution). *Let $f \in V'$, $v_0 \in V'$, and $T > 0$. Then there exists a unique function $v \in C([0,T];V') \cap L^2([0,T];H)$ with $v_t \in L^2([0,T];D(A)')$ and $v(0) = v_0$ and which satisfies (8) in the following sense:*

$$\langle v_t, w\rangle_{D(A)'} + \nu \langle Av, w\rangle_{D(A)'} + (B(\bar{v}, \bar{v}), w) = \langle f, w\rangle_{V'} \tag{9}$$

for every $w \in D(A)$. Moreover, the solution v depends continuously on the initial data, with respect to the $L^\infty([0,T];V')$ norm. Here, (9) is understood in the following sense: for almost every $t_0, t \in [0,T]$ we have

$$\langle v(t), w\rangle_{V'} - \langle v(t_0), w\rangle_{V'} + \nu \int_{t_0}^t (v, Aw) + \int_{t_0}^t (B(\bar{v}(s), \bar{v}(s)), w)\, ds = \int_{t_0}^t \langle f, w\rangle_{V'} ds.$$

3.1. Asymptotic estimates for the long-time dynamics. This section is devoted to establishing appropriate a priori estimates for the long-time dynamics of the solution of (8). In particular, we are required to justify the existence of absorbing balls for the dynamical system induced by (8) in various spaces of functions. This is needed for our proof of the existence of inertial manifolds. Notice that, due to the existence and regularity of the solution stated in Theorem 2, all the estimates provided here are justified. On the other hand, the well-posedness result given in Theorem 2 can be proved by using the Galerkin approximation scheme with a priori estimates derived below. Throughout the following estimates, we assume the forcing $f \in V'$, the initial data $v(0) \in V'$, and thus the corresponding $\bar{v}(0) \in V$.

3.1.1. H^1-estimate for $\bar{v}$. We take the $D(A)'$ action of (8) on $\bar{v}$ and use the identities (5) and (7) to obtain

$$\frac{1}{2}\frac{d}{dt}(|\bar{v}|^2 + \alpha^2\|\bar{v}\|^2) + \nu(\|\bar{v}\|^2 + \alpha^2|A\bar{v}|^2) = \langle f, \bar{v}\rangle. \tag{10}$$

By the Cauchy–Schwarz and Young inequalities, we have

$$|\langle f, \bar{v}\rangle| = |(A^{-1}f, A\bar{v})| \le |A^{-1}f||A\bar{v}| \le \frac{|A^{-1}f|^2}{2\alpha^2\nu} + \frac{\alpha^2\nu}{2}|A\bar{v}|^2.$$

Consequently, we obtain

$$\frac{d}{dt}(|\bar{v}|^2 + \alpha^2\|\bar{v}\|^2) + \nu(\|\bar{v}\|^2 + \alpha^2|A\bar{v}|^2) \le \frac{|A^{-1}f|^2}{\alpha^2\nu}.$$

Applying Poincaré inequality (4), we get

$$\frac{d}{dt}(|\bar{v}|^2 + \alpha^2\|\bar{v}\|^2) + \nu\lambda_1(|\bar{v}|^2 + \alpha^2\|\bar{v}\|^2) \leq \frac{|A^{-1}f|^2}{\alpha^2\nu}.$$

We then use Gronwall's inequality to deduce

$$|\bar{v}(t)|^2 + \alpha^2\|\bar{v}(t)\|^2 \leq e^{-\nu\lambda_1(t-t_0)}(|\bar{v}(t_0)|^2 + \alpha^2\|\bar{v}(t_0)\|^2) + \frac{1-e^{-\nu\lambda_1(t-t_0)}}{\alpha^2\lambda_1\nu^2}|A^{-1}f|^2$$

for all $t \geq t_0 \geq 0$. Therefore

$$\limsup_{t\to\infty}(|\bar{v}(t)|^2 + \alpha^2\|\bar{v}(t)\|^2) \leq \frac{1}{\alpha^2\lambda_1\nu^2}|A^{-1}f|^2.$$

In particular, it follows that

$$\limsup_{t\to\infty}(1+\alpha^2\lambda_1)|\bar{v}(t)|^2 \leq \frac{1}{\alpha^2\lambda_1\nu^2}|A^{-1}f|^2 \quad \text{and} \quad \limsup_{t\to\infty}\alpha^2\|\bar{v}(t)\|^2 \leq \frac{1}{\alpha^2\lambda_1\nu^2}|A^{-1}f|^2.$$

This immediately implies

$$\begin{aligned} &\limsup_{t\to\infty}|\bar{v}(t)| \leq \frac{1}{2}\rho_0 := \left[(1+\alpha^2\lambda_1)\alpha^2\lambda_1\nu^2\right]^{-\frac{1}{2}}|A^{-1}f|, \\ &\limsup_{t\to\infty}\|\bar{v}(t)\| \leq \frac{1}{2}\rho_1 := (\alpha^4\lambda_1\nu^2)^{-\frac{1}{2}}|A^{-1}f|. \end{aligned} \tag{11}$$

Thanks to the above, we conclude that the solution $\bar{v}(t)$, after long enough time, enters a ball in H, centered at the origin, with radius ρ_0. Also, $\bar{v}(t)$ enters a ball in V with radius ρ_1. Notice that the growth of ρ_0 and ρ_1 with respect to the shrinking of ν satisfies $\rho_0 \sim \nu^{-1}$ and $\rho_1 \sim \nu^{-1}$ asymptotically.

3.1.2. H^2-estimate on $\bar{v}$ (L^2-estimate on v). We take the $D(A)'$ action of (8) on $A\bar{v}$ by using (7), and employ the identity $(B(\bar{v},\bar{v}), A\bar{v}) = 0$ (which is valid only in the 2D periodic case; cf. [12, 40]). It follows that

$$\frac{1}{2}\frac{d}{dt}(\|\bar{v}\|^2 + \alpha^2|A\bar{v}|^2) + \nu(|A\bar{v}|^2 + \alpha^2|A^{3/2}\bar{v}|^2) = \langle f, A\bar{v}\rangle.$$

By the Cauchy–Schwarz and Young inequalities, we have

$$|\langle f, A\bar{v}\rangle| = |(A^{-\frac{1}{2}}f, A^{\frac{3}{2}}\bar{v})| \leq \frac{|A^{-1/2}f|^2}{2\alpha^2\nu} + \frac{\alpha^2\nu}{2}|A^{3/2}\bar{v}|^2.$$

As a result, we reach

$$\frac{d}{dt}(\|\bar{v}\|^2 + \alpha^2|A\bar{v}|^2) + \nu(|A\bar{v}|^2 + \alpha^2|A^{3/2}\bar{v}|^2) \leq \frac{|A^{-1/2}f|^2}{\alpha^2\nu}.$$

Applying Poincaré inequality (4) followed by Gronwall's inequality, one has

$$\|\bar{v}(t)\|^2 + \alpha^2|A\bar{v}(t)|^2 \leq e^{-\nu\lambda_1(t-t_0)}(\|\bar{v}(t_0)\|^2 + \alpha^2|A\bar{v}(t_0)|^2) + \frac{1-e^{-\nu\lambda_1(t-t_0)}}{\alpha^2\lambda_1\nu^2}|A^{-1/2}f|^2 \tag{12}$$

for all $t \geq t_0 > 0$. Thus,

$$\limsup_{t\to\infty}(\|\bar{v}(t)\|^2 + \alpha^2|A\bar{v}(t)|^2) \leq \frac{1}{\alpha^2\lambda_1\nu^2}|A^{-1/2}f|^2. \tag{13}$$

In particular, it follows that

$$\limsup_{t\to\infty}\|\bar{v}(t)\| \leq \frac{1}{2}\tilde{\rho}_1 := \left[(1+\alpha^2\lambda_1)\alpha^2\lambda_1\nu^2\right]^{-\frac{1}{2}}|A^{-\frac{1}{2}}f|,$$
$$\limsup_{t\to\infty}|A\bar{v}(t)| \leq \frac{1}{2}\rho_2 := (\alpha^4\lambda_1\nu^2)^{-\frac{1}{2}}|A^{-\frac{1}{2}}f|.$$

The above estimate along with (11) shows that $\|\bar{v}(t)\| \leq \min\{\rho_1, \tilde{\rho}_1\}$ for sufficiently large time t. Also, $\bar{v}(t)$ enters a ball with radius ρ_2 in $D(A)$ after long enough time.

Furthermore, since $v = \bar{v} + \alpha^2 A\bar{v}$, one has

$$\limsup_{t\to\infty}|v(t)| \leq \limsup_{t\to\infty}|\bar{v}(t)| + \alpha^2\limsup_{t\to\infty}|A\bar{v}(t)| \leq \frac{\rho_0+\alpha^2\rho_2}{2}.$$

Thus, after a sufficiently large time, $v(t)$ enters a ball in H with the radius $\rho := \rho_0 + \alpha^2\rho_2$. Also, note that $\rho \sim \nu^{-1}$ asymptotically.

3.2. Existence of an inertial manifold. Define $R(v) := B(\bar{v}, \bar{v})$; then (8) takes the form

$$\frac{dv}{dt} + \nu Av + R(v) = f, \tag{14}$$

where we assume that $f \in V'$. From the energy estimate in subsection 3.1.2, we see that for positive time t one has $\bar{v}(t) \in D(A)$, and thus $v(t) \in H$ for $t > 0$. Moreover, for sufficiently large t, the solution $v(t)$ enters a ball with radius ρ. Since we are concerned with the large-time behavior of solutions, without loss of generality we can assume $v_0 \in H$ throughout the following discussion.

Notice that the nonlinear operator R is locally Lipschitz from H to H. Indeed, let v_1, $v_2 \in H$; then the corresponding $\bar{v}_1$, $\bar{v}_2 \in D(A)$. Furthermore, since $v = \bar{v} + \alpha^2 A\bar{v}$, one has $\bar{v} = (I + \alpha^2 A)^{-1}v$, and thus

$$|A\bar{v}| = |A(I+\alpha^2A)^{-1}v| \leq \frac{1}{\alpha^2}|v|. \tag{15}$$

Then, by using (6), along with the Poincaré inequality and estimate (15), we infer

$$\begin{aligned}
|R(v_1) - R(v_2)| &= |B(\bar{v}_1,\bar{v}_1) - B(\bar{v}_2,\bar{v}_2)| \\
&= |B(\bar{v}_1,\bar{v}_1-\bar{v}_2)| + |B(\bar{v}_1-\bar{v}_2,\bar{v}_2)| \\
&\leq c|\bar{v}_1|^{\frac{1}{2}}\|\bar{v}_1\|^{\frac{1}{2}}\|\bar{v}_1-\bar{v}_2\|^{\frac{1}{2}}|A\bar{v}_1-A\bar{v}_2|^{\frac{1}{2}} + c|\bar{v}_1-\bar{v}_2|^{\frac{1}{2}}\|\bar{v}_1-\bar{v}_2\|^{\frac{1}{2}}\|\bar{v}_2\|^{\frac{1}{2}}|A\bar{v}_2|^{\frac{1}{2}} \\
&\leq c\lambda_1^{-1}(|A\bar{v}_1| + |A\bar{v}_2|)|A\bar{v}_1 - A\bar{v}_2| \\
&\leq c\lambda_1^{-1}\alpha^{-4}(|v_1| + |v_2|)|v_1 - v_2|. \qquad (16)
\end{aligned}$$

As in [12, 21, 22, 40], in order to avoid certain technical difficulties for large values of $|v|$, resulting from the nonlinearity, we truncate the nonlinear term by a smooth cutoff function outside the ball of radius 2ρ in H. Indeed, let $\theta : \mathbb{R}^+ \to [0,1]$ with $\theta(s) = 1$ for $0 \le s \le 1$, $\theta(s) = 0$ for $s \ge 2$, and $|\theta'(s)| \le 2$ for $s \ge 0$. Define $\theta_\rho(s) = \theta(s/\rho)$ for $s \ge 0$. We consider the following "prepared" equation, which is a modification of (14):

$$\frac{dv}{dt} + \nu A v + \theta_\rho(|v|)(R(v) - f) = 0. \tag{17}$$

Notice that (14) and (17) have the same asymptotic behaviors in time and the same dynamics in the neighborhood of the global attractor. This is because we have shown that, for t sufficiently large, $v(t)$ enters a ball in H with radius ρ. On the other hand, the advantage of (17) compared to (14) is that (17) possesses an absorbing *invariant* ball in H. To see this, take the scalar product of (17) with v, and then for $|v| \ge 2\rho$ one has

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \lambda_1 \nu |v|^2 \le \frac{1}{2}\frac{d}{dt}|v|^2 + \nu\|v\|^2 = 0,$$

since $\theta_\rho(|v|) = 0$ for $|v| \ge 2\rho$. It follows that, if $|v_0| > 2\rho$, the orbit of the solution to (17) will converge exponentially to the ball of radius 2ρ in H, while if $|v_0| \le 2\rho$, the solution does not leave this ball.

Furthermore, since $R : H \to H$ is locally Lipschitz, the truncated nonlinearity $F(v) := \theta_\rho(|v|)R(v)$ is *globally* Lipschitz from H to H. To see this, we let $v_1, v_2 \in H$ and calculate for three cases:

(i) If $|v_1| \ge 2\rho$ and $|v_2| \ge 2\rho$, then $F(v_1) = F(v_2) = 0$.

(ii) If $|v_1| \ge 2\rho \ge |v_2|$, then $\theta_\rho(|v_1|) = 0$, and thus

$$\begin{aligned}|F(v_1) - F(v_2)| &= |\theta_\rho(|v_1|)R(v_2) - \theta_\rho(|v_2|)R(v_2)| \\ &\le \frac{2}{\rho}|v_1 - v_2||R(v_2)| \le c\rho\lambda^{-1}\alpha^{-4}|v_1 - v_2|,\end{aligned}$$

by virtue of (16) and the property of θ.

(iii) If $|v_1| \le 2\rho$ and $|v_2| \le 2\rho$, then

$$\begin{aligned}|F(v_1) - F(v_2)| &\le |\theta_\rho(|v_1|)(R(v_1) - R(v_2))| + |R(v_2)(\theta_\rho(|v_1|) - \theta_\rho(|v_2|))| \\ &\le c\rho\lambda_1^{-1}\alpha^{-4}|v_1 - v_2|,\end{aligned}$$

due to (16) and the property of θ.

A summary of these three cases yields

$$|F(v_1) - F(v_2)| \le \mathscr{L}|v_1 - v_2|, \quad \text{where } \mathscr{L} := c\rho\lambda_1^{-1}\alpha^{-4}. \tag{18}$$

Since the nonlinearity of (17) is globally Lipschitz, we shall see that (17) possesses the *strong squeezing property* stated in Proposition 3, provided a certain spectral gap condition is fulfilled. Indeed, for $\gamma > 0$ and $n \in \mathbb{N}$, we define the cone

$$\Gamma_{n,\gamma} := \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H \times H : |Q_n(v_1 - v_2)| \le \gamma |P_n(v_1 - v_2)| \right\}. \tag{19}$$

Here we denote by P_n the orthogonal projection from H onto $\mathrm{span}\{\phi_1, \ldots, \phi_n\}$, where $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of H, and define $Q_n = I - P_n$. The strong squeezing property asserts: if the dynamics of two trajectories starts inside the cone $\Gamma_{n,\gamma}$, then the trajectories stay inside the cone forever, and the higher Fourier modes of the difference are dominated by the lower modes (i.e., *the cone invariance property*); on the other hand, for as long as the two trajectories are outside the cone, then the higher Fourier modes of the difference decay exponentially fast (i.e., *the decay property*). More precisely, we have the following result.

Proposition 3. *Let v_1 and v_2 be two solutions of (17). Then (17) satisfies the following properties:*

(i) *The* cone invariance property: *Assume that n is large enough such that the spectral gap condition $\lambda_{n+1} - \lambda_n > \frac{\mathscr{L}(\gamma+1)^2}{\nu\gamma}$ holds. If $\binom{v_1(t_0)}{v_2(t_0)} \in \Gamma_{n,\gamma}$ for some $t_0 \geq 0$, then $\binom{v_1(t)}{v_2(t)} \in \Gamma_{n,\gamma}$ for all $t \geq t_0$.*

(ii) *The* decay property: *Assume that n is large enough such that $\lambda_{n+1} > \nu^{-1}\mathscr{L}\big(\frac{1}{\gamma}+1\big)$. If $\binom{v_1(t)}{v_2(t)} \notin \Gamma_{n,\gamma}$ for $0 \leq t \leq T$, then*

$$|Q_n(v_1(t) - v_2(t))| \leq |Q_n(v_1(0) - v_2(0))| e^{-b_n t} \qquad \text{for } 0 \leq t \leq T,$$

where $b_n := \nu\lambda_{n+1} - \mathscr{L}\big(\frac{1}{\gamma}+1\big) > 0$.

Proof. See the appendix for the proof. ∎

Notice that the eigenvalues of the operator A satisfy the spectral gap condition:

$$\limsup_{j\to\infty}(\lambda_{j+1} - \lambda_j) = \infty. \tag{20}$$

Indeed, since the eigenvalues of A in the periodic domain are of the form $L^{-2}(k_1^2 + k_2^2)$, the spectral gap condition (20) is available due to a classical result in number theory, given next.

Theorem 4 (Richards [38]). *The sequence $\{\gamma_k = m_1^2 + m_2^2 : m_1, m_2 \in \mathbb{Z} \text{ and } \gamma_{k+1} \geq \gamma_k\}$ has a subsequence $\{\gamma_{k_j}\}$ such that $\gamma_{k_{j+1}} - \gamma_{k_j} \geq \delta \log(\gamma_{k_j})$ for some $\delta > 0$.*

Obviously, (20) implies the required condition in Proposition 3; i.e., there exists $n \in \mathbb{N}$ such that $\lambda_{n+1} - \lambda_n > \frac{4\mathscr{L}}{\nu}$ and $\lambda_{n+1} > \nu^{-1}\mathscr{L}\big(\frac{1}{\gamma}+1\big)$, and thus for such n large enough the strong squeezing property holds for the "prepared" equation (17).

Definition 5 (inertial manifold [21]). *Consider the solution operator $S(t)$ generated by the "prepared" equation (17). A subset $\mathcal{M} \in H$ is called an* initial manifold *for (17) if the following properties are satisfied:*

(i) *$\mathcal{M}$ is a finite-dimensional Lipschitz manifold;*

(ii) *$\mathcal{M}$ is invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$;*

(iii) *$\mathcal{M}$ attracts exponentially all the solutions of (17); i.e.,*

$$\mathrm{dist}(S(t)v_0, \mathcal{M}) \to 0 \quad \text{as } t \to \infty \tag{21}$$

for every $v_0 \in H$, and the rate of decay in (21) is exponential, uniformly for v_0 in bounded sets in H.

Clearly, property (iii) implies that $\mathcal{M}$ contains the global attractor.

It is well known that the strong squeezing property implies the existence of an inertial manifold for dissipative evolution equations. See the references [13, 22, 39, 40]. More precisely, consider a nonlinear evolutionary equation of the type $v_t + Av + N(v) = 0$, where A is a linear, unbounded self-adjoint positive operator, acting in a Hilbert space H, such that A^{-1} is compact, and $N : H \to H$ is a nonlinear operator. Assume that the solution $v(t)$ enters a ball in H with the radius ρ for sufficiently large time t. For $\gamma > 0$ and $n \in \mathbb{N}$ we define the cone $\Gamma_{n,\gamma}$ in (19). Assume that there exists $n \in \mathbb{N}$ such that the prepared equation $v_t+Av+\theta_\rho(|v|)N(v) = 0$ satisfies the strong squeezing property in H (i.e., for any two solutions v_1 and v_2 of the prepared equation, if $\binom{v_1(t_0)}{v_2(t_0)} \in \Gamma_{n,\gamma}$ for some $t_0 \geq 0$, then $\binom{v_1(t)}{v_2(t)} \in \Gamma_{n,\gamma}$ for all $t \geq t_0$; if $\binom{v_1(t)}{v_2(t)} \notin \Gamma_{n,\gamma}$ for $0 \leq t \leq T$, then $|Q_n(v_1(t)-v_2(t))|_H \leq e^{-a_n t}|Q_n(v_1(0)-v_2(0))|_H$ for some $a_n > 0$, $0 \leq t \leq T$); then the dynamics induced by the prepared equation possesses an n-dimensional inertial manifold such that any trajectory approaches the inertial manifold exponentially fast in time with the rate $C(|v_0|_H)e^{-a_n t}$ (e.g., see Theorem 15.5 of [39] for a proof). Moreover, the following *exponential tracking* property holds for this inertial manifold: for any $v_0 \in H$ there exist a time $\tau \geq 0$ and a solution $S(t)\varphi_0$ on the inertial manifold such that $|S(t+\tau)v_0 - S(t)\varphi_0|_H \leq Ce^{-a_n t}$, where the constant C depends on $|S(\tau)v_0|_H$ and $|\varphi_0|_H$ (see Theorem 5.2 of [22] for a proof). As a result, since we have shown that the strong squeezing property holds for (17), provided that n is large enough, we obtain the following result for the simplified Bardina model.

Theorem 6. *The "prepared" equation* (17) *of the simplified Bardina model possesses an n-dimensional inertial manifold $\mathcal{M}$ in H; i.e., the solution $S(t)v_0$ of* (17) *approaches the invariant Lipschitz manifold $\mathcal{M}$ exponentially. Furthermore, the following exponential tracking property holds: for any $v_0 \in H$ there exist a time $\tau \geq 0$ and a solution $S(t)\varphi_0$ on the inertial manifold $\mathcal{M}$ such that*

$$|S(t+\tau)v_0 - S(t)\varphi_0| \leq Ce^{-b_n t},$$

where b_n is defined in Proposition 3 *and the constant C depends on $|S(\tau)v_0|$ and $|\varphi_0|$.*

4. The modified-Leray-α model.

This section is devoted to proving the existence of an inertial manifold for the modified-Leray-α model (3). Applying the Helmholtz–Leray orthogonal projection P_σ to (3), we obtain the following equivalent functional differential equation:

$$\begin{cases} u_t + \nu Au + B(u,\bar{u}) = f, \\ u = \bar{u} + \alpha^2 A\bar{u}, \\ u(x,0) = u_0(x). \end{cases} \tag{22}$$

An analytical study of the modified-Leray-α model has been presented in [24]. Specifically, it was shown that (22) is globally well-posed in three dimensions. In addition, an upper bound for the dimension of its global attractor and analysis of the energy spectrum were established. The proof of global well-posedness in two dimensions is very similar, so we just state the result and omit its proof.

Theorem 7 (regular solution). *Let $f \in H$, $u_0 \in V'$, and $T > 0$. Then there exists a unique function $u \in C([0,T];V') \cap L^2([0,T];H)$ with $u_t \in L^2([0,T];D(A)')$ and $u(0) = u_0$, and which satisfies* (22) *in the following sense:*

$$\left\langle \frac{du}{dt}, w \right\rangle_{D(A)'} + \nu \langle Au, w\rangle_{D(A)'} + (B(u,\bar{u}), w) = \langle f, w\rangle_{V'} \tag{23}$$

for every $w \in D(A)$. Moreover, the solution u depends continuously on the initial data with respect to the $L^\infty([0,T];V')$ norm. Here, (23) is understood in the following sense: for almost everywhere $t_0, t \in [0,T]$ we have

$$\langle u(t), w\rangle_{V'} - \langle u(t_0), w\rangle_{V'} + \nu \int_{t_0}^{t} (u, Aw) + \int_{t_0}^{t} (B(u(s), \bar{u}(s)), w)\, ds = \int_{t_0}^{t} \langle f, w\rangle_{V'} ds.$$

4.1. Asymptotic estimates for the long-time dynamics. In order to prove the existence of an inertial manifold, it is required to establish appropriate a priori estimates on the long-time dynamics of the solution. In particular, we are required to find absorbing balls for the dynamical system induced by the (22) in various spaces of functions. During our estimates, $u_0 \in V'$ and $f \in H$.

4.1.1. H^1-estimate on $\bar{u}$. Taking the $D(A)'$ action of the (22) on $\bar{u}$ by using the fact $(B(u,\bar{u}),\bar{u}) = 0$ and (7), we obtain

$$\frac{1}{2}\frac{d}{dt}(|\bar{u}|^2 + \alpha^2\|\bar{u}\|^2) + \nu(\|\bar{u}\|^2 + \alpha^2|A\bar{u}|^2) = (f, \bar{u}). \tag{24}$$

Notice that the energy identity (24) is almost identical to (10) from the analysis of the simplified Bardina model. Therefore, we can adopt the estimate in subsection 3.1.1 to conclude

$$\limsup_{t\to\infty} |\bar{u}(t)| \le \frac{1}{2}\rho_0 := \left[(1+\alpha^2\lambda_1)\alpha^2\lambda_1\nu^2\right]^{-\frac{1}{2}} |A^{-1}f|,$$
$$\limsup_{t\to\infty} \|\bar{u}(t)\| \le \frac{1}{2}\rho_1 := (\alpha^4\lambda_1\nu^2)^{-\frac{1}{2}}|A^{-1}f|.$$

From this, we conclude that the solution $\bar{u}(t)$, after a sufficiently large time, enters a ball in H with radius ρ_0 and also enters a ball in V with radius ρ_1. In addition the growth of the radii ρ_0 and ρ_1 with respect to the shrinking of the viscosity ν satisfies $\rho_0 \sim \nu^{-1}$ and $\rho_1 \sim \nu^{-1}$.

4.1.2. L^2-estimate on u (H^2-estimate on $\bar{u}$). By taking the $D(A)'$ action of (22) on u and using (7), we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 + (B(u,\bar{u}), u) = (f, u).$$

Recall from subsection 3.1.2 that when we derived the L^2-estimate on v (H^2-estimate on $\bar{v}$) for the simplified Bardina model, we used the identity $(B(\bar{v},\bar{v}), A\bar{v}) = 0$ (in the periodic 2D case) to eliminate the nonlinearity. On the other hand, for the NSE, the L^2-estimate is fairly easy, since $(B(u,u),u) = 0$. However, under the current situation, the nonlinear term $(B(u,\bar{u}),u)$ does not vanish, which causes the estimate to be slightly more involved. Indeed, by using Hölder's inequality, the Ladyzhenskaya inequality $|u|_{L^4} \le c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}$, and the Young's inequality, we infer

$$|(B(u,\bar{u}),u)| \le |u|_{L^4}^2\|\bar{u}\| \le c|u|\|u\|\|\bar{u}\| \le \frac{\nu}{4}\|u\|^2 + \frac{c}{\nu}|u|^2\|\bar{u}\|^2.$$

Also, $|(f,u)| = |(A^{-\frac{1}{2}}f, A^{\frac{1}{2}}u)| \le |A^{-\frac{1}{2}}f|\|u\| \le \frac{\nu}{4}\|u\|^2 + \frac{1}{\nu}|A^{-\frac{1}{2}}f|^2$. Combining the above estimates, we obtain

$$\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{c}{\nu}|u|^2\|\bar{u}\|^2 + \frac{2}{\nu}|A^{-\frac{1}{2}}f|^2.$$

In subsection 4.1.1 we have shown that there exists $t_1 > 0$ such that $|\bar{u}(t)| \le \rho_0$ and $\|\bar{u}(t)\| \le \rho_1$ provided $t \ge t_1$. As a result,

$$\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{c}{\nu}\rho_1^2|u|^2 + \frac{2}{\nu}|A^{-\frac{1}{2}}f|^2 \quad \text{for all } t \ge t_1. \tag{25}$$

We attempt to derive a uniform bound for $|u(t)|$. To this end, we integrate between s and $t + \frac{1}{\nu\lambda_1}$ for $t_1 \le t \le s \le t + \frac{1}{\nu\lambda_1}$:

$$\left|u\left(t + \frac{1}{\nu\lambda_1}\right)\right|^2 \le |u(s)|^2 + \frac{c}{\nu}\rho_1^2 \int_t^{t+\frac{1}{\nu\lambda_1}} |u(\tau)|^2 d\tau + \frac{2}{\nu^2\lambda_1}|A^{-\frac{1}{2}}f|^2.$$

Then, integrating with respect to s from t to $t + \frac{1}{\nu\lambda_1}$ gives

$$\frac{1}{\nu\lambda_1}\left|u\left(t + \frac{1}{\nu\lambda_1}\right)\right|^2 \le \left(\frac{c}{\nu^2\lambda_1}\rho_1^2 + 1\right) \int_t^{t+\frac{1}{\nu\lambda_1}} |u(s)|^2 ds + \frac{2}{\nu^3\lambda_1^2}|A^{-\frac{1}{2}}f|^2 \quad \text{for all } t \ge t_1. \tag{26}$$

In order to control the right-hand side, we should obtain a bound on $\int_t^{t+\frac{1}{\nu\lambda_1}} |u(s)|^2 ds$. To this end, we deduce from (24) by using the Cauchy–Schwarz and Young inequalities,

$$\frac{d}{dt}(|\bar{u}|^2 + \alpha^2||\bar{u}||^2) + \nu(||\bar{u}||^2 + \alpha^2|A\bar{u}|^2) \le \frac{|A^{-1}f|^2}{\alpha^2\nu}.$$

Integrating the above inequality from t to $t + \frac{1}{\nu\lambda_1}$ yields

$$\begin{aligned}\nu\alpha^2 \int_t^{t+\frac{1}{\nu\lambda_1}} |A\bar{u}(s)|^2 ds &\le |\bar{u}(t)|^2 + \alpha^2||\bar{u}(t)||^2 + \frac{|A^{-1}f|^2}{\alpha^2\nu^2\lambda_1} \\ &\le \rho_0^2 + \alpha^2\rho_1^2 + \frac{|A^{-1}f|^2}{\alpha^2\nu^2\lambda_1} \quad \text{for } t \ge t_1,\end{aligned}$$

where we have used the fact that $|\bar{u}(t)| \le \rho_0$ and $\|\bar{u}(t)\| \le \rho_1$ for $t \ge t_1$.

By definition $u = \bar{u} + \alpha^2 A\bar{u}$, it follows that $|u|^2 \le 2(|\bar{u}|^2 + \alpha^4|A\bar{u}|^2) \le 2\big(\frac{1}{\lambda_1^2} + \alpha^4\big)|A\bar{u}|^2$ due to the Poincaré inequality. Consequently, for $t \ge t_1$, one has

$$\begin{aligned}\int_t^{t+\frac{1}{\nu\lambda_1}} |u(s)|^2 ds &\le 2\left(\frac{1}{\lambda_1^2} + \alpha^4\right) \int_t^{t+\frac{1}{\nu\lambda_1}} |A\bar{u}(s)|^2 ds \\ &\le C_0 := \left(\frac{1}{\lambda_1^2} + \alpha^4\right) \frac{2}{\nu\alpha^2}\left(\rho_0^2 + \alpha^2\rho_1^2 + \frac{|A^{-1}f|^2}{\alpha^2\nu^2\lambda_1}\right).\end{aligned} \tag{27}$$

Substituting (27) into (26), we conclude

$$\left|u\left(t+\frac{1}{\nu\lambda_1}\right)\right|^2 \le \rho_3^2 := \left(\frac{c}{\nu}\rho_1^2+\nu\lambda_1\right)C_0+\frac{2}{\nu^2\lambda_1}|A^{-\frac{1}{2}}f|^2 \quad \text{for } t\ge t_1.$$

This indicates that, for $t \ge t_1 + \frac{1}{\nu\lambda_1}$, the solution $u(t)$ enters a ball in H with the radius ρ_3.

Furthermore, the growth of the radius ρ_3 with respect to the shrinking of the viscosity ν satisfies $\rho_3 \sim \nu^{-3}$.

4.1.3. H^1-estimate on u. We take the $D(A)'$ action of (22) on Au. It follows from (7) that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\nu|Au|^2+(B(u,\bar{u}),Au)=(f,Au).$$

By using the Cauchy–Schwarz and Young inequalities, one has

$$\frac{d}{dt}\|u\|^2+\nu|Au|^2 \le \frac{2}{\nu}(|B(u,\bar{u})|^2+|f|^2).$$

Recall that we have shown that $\|\bar{u}(t)\| \le \rho_1$ for $t \ge t_1$, as well as $|u(t)| \le \rho_3$ for $t \ge t_1 + \frac{1}{\nu\lambda_1}$. Therefore, by employing (6) along with (15), we deduce

$$|B(u,\bar{u})| \le c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\bar{u}\|^{\frac{1}{2}}|A\bar{u}|^{\frac{1}{2}} \le \frac{c}{\alpha}|u|\|u\|^{\frac{1}{2}}\|\bar{u}\|^{\frac{1}{2}} \le \frac{c}{\alpha}\rho_3\rho_1^{\frac{1}{2}}\|u\|^{\frac{1}{2}} \quad \text{for } t \ge t_1+\frac{1}{\nu\lambda_1}.$$

As a result, for $t \ge t_1 + \frac{1}{\nu\lambda_1}$,

$$\frac{d}{dt}\|u\|^2 \le \frac{c}{\nu\alpha^2}\rho_3^2\rho_1\|u\|+\frac{2}{\nu}|f|^2.$$

To obtain a uniform bound for $\|u(t)\|$, we integrate between s and $t+\frac{1}{\nu\lambda_1}$, for $t_1+\frac{1}{\nu\lambda_1} \le t \le s \le t+\frac{1}{\nu\lambda_1}$:

$$\left\|u\left(t+\frac{1}{\nu\lambda_1}\right)\right\|^2 \le \|u(s)\|^2+\frac{c}{\nu\alpha^2}\rho_3^2\rho_1\int_t^{t+\frac{1}{\nu\lambda_1}}\|u(\tau)\|d\tau+\frac{2}{\nu^2\lambda_1}|f|^2.$$

Then, using the Cauchy–Schwarz inequality and integrating with respect to s between t and $t+\frac{1}{\nu\lambda_1}$ yields

$$\frac{1}{\nu\lambda_1}\left\|u\left(t+\frac{1}{\nu\lambda_1}\right)\right\|^2 \le \int_t^{t+\frac{1}{\nu\lambda_1}}\|u(s)\|^2ds+\frac{c}{\nu^{\frac{5}{2}}\alpha^2\lambda_1^{\frac{3}{2}}}\rho_3^2\rho_1\left(\int_t^{t+\frac{1}{\nu\lambda_1}}\|u(s)\|^2ds\right)^{\frac{1}{2}}+\frac{2}{\nu^3\lambda_1^2}|f|^2$$

for $t \ge t_1+\frac{1}{\nu\lambda_1}$. Now we ought to find a bound for $\int_t^{t+\frac{1}{\nu\lambda_1}}\|u(s)\|^2ds$. Indeed, integrating (25) from t to $t+\frac{1}{\nu\lambda_1}$ for $t \ge t_1+\frac{1}{\nu\lambda_1}$ gives

$$\begin{aligned}\int_t^{t+\frac{1}{\nu\lambda_1}}\|u(s)\|^2ds &\le \frac{1}{\nu}\left(|u(t)|^2+\frac{c}{\nu}\rho_1^2\int_t^{t+\frac{1}{\nu\lambda_1}}|u(s)|^2ds+\frac{2}{\nu^2\lambda_1}|A^{-\frac{1}{2}}f|^2\right)\\ &\le C_1 := \frac{1}{\nu}\left(\rho_3^2+\frac{c}{\nu}\rho_1^2C_0+\frac{2}{\nu^2\lambda_1}|A^{-\frac{1}{2}}f|^2\right),\end{aligned}$$

where we have used (27) and the fact that $|u(t)| \le \rho_3$ provided $t \ge t_1 + \frac{1}{\nu\lambda_1}$.

Finally, we conclude

$$\|u(t)\|^2 \le \tilde{\rho}^2 := \nu\lambda_1 C_1 + \frac{c}{\nu^{\frac{3}{2}}\alpha^2\lambda_1^{\frac{1}{2}}}\rho_3^2\rho_1 C_1^{\frac{1}{2}} + \frac{2}{\nu^2\lambda_1}|f|^2 \quad \text{for } t \ge t_1 + \frac{2}{\nu\lambda_1}.$$

This shows that the solution $u(t)$ enters a ball in V of radius $\tilde{\rho}$ for $t \ge t_1 + \frac{2}{\nu\lambda_1}$.

Also, recall $\rho_0 \sim \nu^{-1}$, $\rho_1 \sim \nu^{-1}$, $\rho_3 \sim \nu^{-3}$; then by (27) one has $C_0 \sim \nu^{-3}$, and thus we see that $C_1 \sim \nu^{-7}$. Hence, $\tilde{\rho} \sim \nu^{-6}$.

4.2. Existence of an inertial manifold. From energy estimates established in section 4.1, we see that for positive time t one has $u(t) \in V$ because of the parabolic nature of the equation, and for sufficiently large time $t \ge t_1 + \frac{2}{\nu\lambda_1}$ the solution $u(t)$ enters a ball in V of radius $\tilde{\rho}$. So, without loss of generality, as far as the inertial manifold is concerned, which is a long-time behavior, we assume the initial data $u_0 \in V$.

We set $\mathcal{R}(u) := B(u, \bar{u})$. Then (22) takes the form

$$u_t + \nu A u + \mathcal{R}(u) = f. \tag{28}$$

Recall that the nonlinear term $B(\bar{v}, \bar{v}) = P_\sigma(\bar{v} \cdot \nabla)\bar{v}$ in the simplified Bardina model (1) is locally Lipschitz from H to H, which is a condition for (1) possessing an inertial manifold. However, $\mathcal{R}(u)$ does not have this property, since it is not a mapping from H to H. However, we will be able to show that $\mathcal{R}$ is locally Lipschitz continuous from V to V. To see this, we calculate

$$\begin{aligned}
\|\mathcal{R}(u)\| = |A^{\frac{1}{2}}B(u, \bar{u})| &\le |B(A^{\frac{1}{2}}u, \bar{u})| + |B(u, A^{\frac{1}{2}}\bar{u})| \\
&\le \|u\||\nabla\bar{u}|_{L^\infty} + c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|A\bar{u}|^{\frac{1}{2}}|A^{\frac{3}{2}}\bar{u}|^{\frac{1}{2}} \\
&\le c\|u\|\|\bar{u}\|^{\frac{1}{2}}|A^{\frac{3}{2}}\bar{u}|^{\frac{1}{2}} + c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|A\bar{u}|^{\frac{1}{2}}|A^{\frac{3}{2}}\bar{u}|^{\frac{1}{2}} \\
&\le c\lambda_1^{-\frac{1}{2}}\|u\||A^{\frac{3}{2}}\bar{u}| \\
&\le c\lambda_1^{-\frac{1}{2}}\alpha^{-2}\|u\|^2.
\end{aligned} \tag{29}$$

Note that, throughout the above calculation, we have employed (6), and Agmon's inequality [1, 2] in two dimensions—$|\phi|_{L^\infty} \le c|\phi|^{\frac{1}{2}}|A\phi|^{\frac{1}{2}}$—as well as (15), where c is a positive constant.

This shows that $\mathcal{R}$ is a mapping from V to V. By similar computation, we deduce, for u_1, $u_2 \in V$,

$$\|\mathcal{R}(u_1) - \mathcal{R}(u_2)\| \le c\lambda_1^{-\frac{1}{2}}\alpha^{-2}(\|u_1\| + \|u_2\|)\|u_1 - u_2\|; \tag{30}$$

that is, $\mathcal{R} : V \to V$ is locally Lipschitz continuous.

Recall that in subsection 4.1.3 we have shown that $|u(t)| \le \tilde{\rho}$ for sufficiently large time $t \ge t_1 + \frac{2}{\nu\lambda_1}$. As in [21, 22], in order to avoid certain technical difficulties for large values of $\|u\|$, resulting from the nonlinearity, we truncate the nonlinear term outside the ball of radius $2\tilde{\rho}$ in V by a smooth cutoff function $\theta : \mathbb{R}^+ \to [0, 1]$ with $\theta(s) = 1$ for $0 \le s \le 1$, $\theta(s) = 0$ for

$s \geq 2$, and $|\theta'(s)| \leq 2$ for $s \geq 0$. Define $\theta_{\tilde{\rho}}(s) = \theta(s/\tilde{\rho})$ for $s \geq 0$. We consider the following "prepared" equation, which is a modification of (28):

$$u_t + \nu A u + \theta_{\tilde{\rho}}(\|u\|)(\mathcal{R}(u) - f) = 0. \tag{31}$$

Since $\mathcal{R} : V \to V$ is locally Lipschitz, by a calculation similar to that in subsection 3.2, it can be shown that the truncated nonlinearity $\mathcal{F}(u) := \theta_{\tilde{\rho}}(\|u\|)\mathcal{R}(u)$ is globally Lipschitz continuous with Lipschitz constant $\mathcal{L} := c\tilde{\rho}\lambda_1^{-\frac{1}{2}}\alpha^{-2}$.

Now, for $\gamma > 0$ and $N \in \mathbb{N}$, we define the cone in the product space $V \times V$:

$$\tilde{\Gamma}_{N,\gamma} := \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in V \times V : \|Q_N(u_1 - u_2)\| \leq \gamma \|P_N(u_1 - u_2)\| \right\}.$$

The following result states that (31) possesses the *strong squeezing property*.

Proposition 8. *Let u_1 and u_2 be two solutions of (31). Then (31) satisfies the following properties:*

(i) *The* cone invariance property: *Assume that N is large enough such that the spectral gap condition $\lambda_{N+1} - \lambda_N > \frac{\mathcal{L}(\gamma+1)^2}{\nu\gamma}$ holds. If $\binom{u_1(t_0)}{u_2(t_0)} \in \tilde{\Gamma}_{N,\gamma}$ for some $t_0 \geq 0$, then $\binom{u_1(t)}{u_2(t)} \in \tilde{\Gamma}_{N,\gamma}$ for all $t \geq t_0$.*

(ii) *The* decay property: *Assume N is sufficiently large such that $\lambda_{N+1} > \nu^{-1}\mathcal{L}\big(\frac{1}{\gamma} + 1\big)$. If $\binom{u_1(t)}{u_2(t)} \notin \tilde{\Gamma}_{N,\gamma}$ for $0 \leq t \leq T$, then*

$$\|Q_N(u_1(t) - u_2(t))\| \leq \|Q_N(u_1(0) - u_2(0))\| e^{-\beta_N t} \quad \text{for } 0 \leq t \leq T,$$

where $\beta_N := \nu\lambda_{N+1} - \mathcal{L}\big(\frac{1}{\gamma} + 1\big) > 0$.

Proof. See the appendix for the proof. ∎

Note that the spectral gap condition is satisfied for sufficiently large N, by virtue of Theorem 4. Consequently the strong squeezing property holds for (31), provided that N is large enough. Recall that we have mentioned in section 3.2 that the strong squeezing property implies the existence of an N-dimensional inertial manifold (see, e.g., Theorem 15.5 of [39]), as well as the exponential tracking property for this inertial manifold (see Theorem 5.2 of [22]); thus we have the following result.

Theorem 9. *The "prepared" equation (31) of the modified-Leray-α model possesses an N-dimensional inertial manifold $\mathfrak{M}$ in V; i.e., the solution $S(t)u_0$ of (31) approaches the invariant Lipschitz manifold $\mathfrak{M}$ exponentially in V. Furthermore, the following exponential tracking property holds: for any $u_0 \in V$ there exist a time $\tau \geq 0$ and a solution $S(t)\varphi_0$ on the inertial manifold $\mathfrak{M}$ such that*

$$\|S(t+\tau)u_0 - S(t)\varphi_0\| \leq Ce^{-\beta_N t},$$

where β_N is defined in Proposition 8 and the constant C depends on $\|S(\tau)u_0\|$ and $\|\varphi_0\|$.

Remark 1. Concerning the Leray-α model (2), the nonlinearity is $(\bar{w} \cdot \nabla)w$, and clearly there is a loss of a derivative. It can be shown that the operator $\tilde{R}(v) := B(\bar{v}, v) = P_\sigma(\bar{v} \cdot \nabla)v$ is Lipschitz continuous from V to H in two dimensions. As far as the inertial manifold is concerned, this produces a difficulty similar to what we face for the 2D NSE. Indeed, under such a scenario, using the classical theory, the existence of an inertial manifold requires a stronger gap condition: $\lambda_{j+1}^{\frac{1}{2}} - \lambda_j^{\frac{1}{2}}$ must be sufficiently big, which holds only for very large viscosity ν (see, e.g., [39]). But our main interest lies in fluid flow with small viscosity, which is the situation when turbulence occurs, so a result valid for only large ν is of no account.

5. Appendix.

We present the proof of Propositions 3 and 8 for the sake of completion. Since the proofs of these two propositions are similar, we show only that for Proposition 8.

Proof. The method of the proof is standard (see, e.g., [22, 40]). Assume u_1 and u_2 are two solutions of (31). To show the cone invariance property (i), it is sufficient to show that $\binom{u_1(t)}{u_2(t)}$ cannot pass through the boundary of the cone if the dynamics starts inside the cone. More precisely, we shall show that $\frac{d}{dt}(\|Q_N(u_1(t)-u_2(t))\| - \gamma\|P_N(u_1(t)-u_2(t))\|) < 0$ whenever $\binom{u_1(t)}{u_2(t)} \in \partial\tilde{\Gamma}_{N,\gamma}$, where $\partial\tilde{\Gamma}_{N,\gamma}$ stands for the boundary of the cone $\tilde{\Gamma}_{N,\gamma}$.

Recall $\mathcal{F}(u) = \theta_{\tilde{\rho}}(\|u\|)\mathcal{R}(u)$. Then by (31),

$$\partial_t(u_1-u_2) + \nu A(u_1-u_2) + \mathcal{F}(u_1) - \mathcal{F}(u_2) = 0.$$

By setting $p = P_N(u_1-u_2)$ and $q = Q_N(u_1-u_2)$, we obtain

$$p_t + \nu Ap + P_N(\mathcal{F}(u_1) - \mathcal{F}(u_2)) = 0, \tag{32}$$

$$q_t + \nu Aq + Q_N(\mathcal{F}(u_1) - \mathcal{F}(u_2)) = 0. \tag{33}$$

We take the scalar product of (32) with Ap,

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 + \nu|Ap|^2 + (P_N(\mathcal{F}(u_1) - \mathcal{F}(u_2)), Ap) = 0.$$

Thus by the global Lipschitz continuity of $\mathcal{F}$, we have

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 \geq -\nu\lambda_N\|p\|^2 - \|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|\|p\| \geq -\nu\lambda_N\|p\|^2 - \mathcal{L}\|u_1-u_2\|\|p\|. \tag{34}$$

Without loss of generality, we can assume $\|p(t)\| > 0$. (Otherwise, if $\|p(t^*)\| = 0$ for some t^*, then since we consider the boundary of the cone, we can assume $\|q(t^*)\| = \gamma\|p(t^*)\| = 0$, and thus $u_1(t^*) = u_2(t^*)$. By the uniqueness of solutions, we obtain $u_1(t) = u_2(t)$ for all $t \geq t^*$, and the cone invariance property follows.) Now we can divide both sides of (34) by $\|p(t)\|$, so

$$\frac{d}{dt}\|p\| \geq -\nu\lambda_N\|p\| - \mathcal{L}\|u_1-u_2\|. \tag{35}$$

Analogously, by taking the scalar product of (33) with Aq, we can deduce

$$\frac{d}{dt}\|q\| \leq -\nu\lambda_{N+1}\|q\| + \mathcal{L}\|u_1-u_2\|. \tag{36}$$

Multiplying (35) with γ and subtracting the result from (36), we infer, by using the fact $p+q = u_1-u_2$,

$$\frac{d}{dt}(\|q\| - \gamma\|p\|) \leq \nu(\lambda_N\gamma\|p\| - \lambda_{N+1}\|q\|) + \mathcal{L}(\gamma+1)(\|p\|+\|q\|).$$

So whenever $\|q(t)\| = \gamma\|p(t)\|$, i.e., $\binom{u_1(t)}{u_2(t)} \in \partial\tilde{\Gamma}_{N,\gamma}$, we have

$$\frac{d}{dt}(\|q\| - \gamma\|p\|) \leq \left(\nu(\lambda_N - \lambda_{N+1}) + \mathcal{L}\frac{(\gamma+1)^2}{\gamma}\right)\|q\| < 0,$$

due to our assumption $\lambda_{N+1} - \lambda_N > \frac{\mathcal{L}(\gamma+1)^2}{\nu\gamma}$.

To show the decay property (ii), we assume $\binom{u_1(t)}{u_2(t)} \notin \tilde{\Gamma}_{N,\gamma}$ for $0 \leq t \leq T$; then $\|q(t)\| > \gamma\|p(t)\|$ for $0 \leq t \leq T$, and we see from (36) that

$$\frac{d}{dt}\|q\| \leq -\nu\lambda_{N+1}\|q\| + \mathcal{L}(\|p\| + \|q\|) \leq -\left[\nu\lambda_{N+1} - \mathcal{L}\left(\frac{1}{\gamma}+1\right)\right]\|q\| = -\beta_N\|q\|$$

for $0 \leq t \leq T$, where $\beta_N := \nu\lambda_{N+1} - \mathcal{L}\left(\frac{1}{\gamma}+1\right)$. By Gronwall's inequality, one has

$$\|q(t)\| \leq e^{-\beta_N t}\|q(0)\| \quad \text{for } \ 0 \leq t \leq T. \qquad \blacksquare$$

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