SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH DAMPING AND SUPERCRITICAL BOUNDARY AND INTERIOR SOURCES

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ABSTRACT. We consider the local and global well-posedness of the coupled nonlinear wave equations

$$u_{tt} - \Delta u + g_1(u_t) = f_1(u,v),$$
$$v_{tt} - \Delta v + g_2(v_t) = f_2(u,v)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with Robin and Dirichlét boundary conditions on u and v respectively. The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ have supercritical exponents representing strong sources, while $g_1(u_t)$ and $g_2(v_t)$ act as damping. In addition, the boundary condition also contains a nonlinear source and a damping term. By employing nonlinear semigroups and the theory of monotone operators, we obtain several results on the existence of local and global weak solutions, and uniqueness of weak solutions. Moreover, we prove that such unique solutions depend continuously on the initial data.

1. INTRODUCTION AND MAIN RESULTS

1.1. **The model.** In this article, we study a system of coupled nonlinear wave equations which features two competing forces. One force is damping and the other is a strong source. Of central interest is the relationship of the source and damping terms to the behavior of solutions.

In order to simplify the exposition, we restrict our analysis to the physically more relevant case when $\Omega \subset \mathbb{R}^3$. Our results extend very easily to bounded domains in $\mathbb{R}^n$, by accounting for the corresponding Sobolev imbeddings, and accordingly adjusting the conditions imposed on the parameters. Therefore, throughout the paper we assume that Ω is a bounded, open, and connected nonempty set in $\mathbb{R}^3$ with a smooth boundary $\Gamma = \partial\Omega$.

We study the local and global well-posedness of the following initial boundary value problem:

$$(1.1) \quad \begin{cases} u_{tt} - \Delta u + g_1(u_t) = f_1(u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u,v) & \text{in } \Omega \times (0,T), \\ \partial_\nu u + u + g(u_t) = h(u) & \text{on } \Gamma \times (0,T), \\ v = 0 & \text{on } \Gamma \times (0,T), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H^1_0(\Omega), v_t(0) = v_1 \in L^2(\Omega), \end{cases}$$

where the nonlinearities $f_1(u,v)$, $f_2(u,v)$ and $h(u)$ are supercritical interior and boundary sources, and the damping functions g_1, g_2 and g are arbitrary continuous monotone increasing graphs vanishing at the origin.

The source-damping interaction in (1.1) encompasses a broad class of problems in quantum field theory and certain mechanical applications (Jörgens [18] and Segal [38]). A related model to (1.1) is the Reissner-Mindlin plate equations (see for instance Ch. 3 in [21]), which consist of three coupled PDE's: a wave equation and two wave-like equations, where each equation is influenced by nonlinear damping and source terms. It is worth noting that *nondissipative* "energy-building" sources, especially those on the boundary, arise when one considers a wave equation being coupled with other types of dynamics, such as structure-acoustic or fluid-structure interaction models (Lasiecka [25]). In light of these applications we are mainly interested in higher-order nonlinearities, as described in the following assumption.

Assumption 1.1.

- **Damping:** *g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0) = g_2(0) = g(0) = 0$. In addition, the following growth conditions at infinity hold: there exist positive constants α and β such that, for $|s| \geq 1$,*

$$\alpha|s|^{m+1} \leq g_1(s)s \leq \beta|s|^{m+1}, \quad with \ m \geq 1,$$
$$\alpha|s|^{r+1} \leq g_2(s)s \leq \beta|s|^{r+1}, \quad with \ r \geq 1,$$
$$\alpha|s|^{q+1} \leq g(s)s \leq \beta|s|^{q+1}, \quad with \ q \geq 1.$$

- **Interior sources:** *$f_j(u,v) \in C^1(\mathbb{R}^2)$ such that*

$$|\nabla f_j(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1), \quad j = 1, 2, \quad with \ 1 \leq p < 6.$$

- **Boundary source:** *$h \in C^1(\mathbb{R})$ such that*

$$|h'(s)| \leq C(|s|^{k-1} + 1), \quad with \ 1 \leq k < 4.$$

- **Parameters:** $\max\{p\frac{m+1}{m}, p\frac{r+1}{r}\} < 6, \quad k\frac{q+1}{q} < 4.$

Let us note here that in view of the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (in 3D), each of the Nemytski operators $f_j(u,v)$ is locally Lipschitz continuous from $H^1(\Omega) \times H^1(\Omega)$ into $L^2(\Omega)$ for the values $1 \leq p \leq 3$. Hence, when the exponent of the sources p lies in $1 \leq p < 3$, we call the source **subcritical**, and **critical** if $p = 3$. For the values $3 < p \leq 5$ the source is called **supercritical**, and in this case the operator $f_j(u,v)$ is **not** locally Lipschitz continuous from $H^1(\Omega) \times H^1(\Omega)$ into $L^2(\Omega)$. However, for $3 < p \leq 5$, the potential energy induced by the source is well defined in the finite energy space. When $5 < p < 6$ the source is called **super-supercritical**. In this case, the potential energy may not be defined in the finite energy space and the problem itself is no longer within the framework of potential well theory (see for instance [2, 27, 28, 43, 44]).

A benchmark system, which is a special case of (1.1), is the following well-known polynomially damped system studied extensively in the literature (see for instance

[1, 2, 32, 33]):

$$(1.2) \qquad \begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u, v) & \text{in } \Omega \times (0, T), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u, v) & \text{in } \Omega \times (0, T), \end{cases}$$

where the sources f_1, f_2 are very specific. Namely, $f_1(u, v) = \partial_u F(u, v)$ and $f_2(u, v) = \partial_v F(u, v)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the C^1-function given by

$$F(u, v) = a|u + v|^{p+1} + 2b|uv|^{\frac{p+1}{2}},$$

where $p \geq 3$, $a > 1$ and $b > 0$. Systems of nonlinear wave equations such as (1.2) go back to Reed [35], who proposed a similar system in three space dimensions but without the presence of damping. Indeed, recently in [1] and later in [2] the authors studied system (1.2) with Dirichlét boundary conditions on both u and v where the exponent of the source was restricted to be *critical* ($p = 3$ in 3D). We note here that the functions f_1 and f_2 in (1.2) satisfy Assumption 1.1, even for the values $3 \leq p < 6$, and so our work extends and refines the results in [1], on one hand by allowing a larger class of sources (other than those in (1.2)) and having a larger range of exponents of sources, $p > 3$. On the other hand, system (1.1) has a Robin boundary condition which also features nonlinear damping and a source term. In particular, the Robin boundary condition, in combination with the interior damping, creates serious technical difficulties in the analysis (for more details, see Subsection 2.1).

In studying systems such as (1.2) or the more general system (1.1), several difficulties arise due to the coupling. On one hand, establishing blow up results for systems of wave equations (not just global nonexistence results which don't require local solvability) is known to be more subtle than the scalar case. Additional challenges stem from the fact that in many physical systems, such as (1.2), the sources are not necessarily C^2-functions, even when $3 < p \leq 5$. In such a case, uniqueness of solutions becomes problematic, and this particular issue will be addressed in this paper.

In recent years, wave equations under the influence of nonlinear damping and sources have generated considerable interest. If the sources are at most critical, i.e., $p \leq 3$ and $k \leq 2$, many authors have successfully studied such equations by using Galerkin approximations or standard fixed point theorems (see for example [1–3, 14, 29, 32–34]). Also, for other related work on hyperbolic problems, we refer the reader to [11, 13, 17, 20, 23, 24, 26, 31, 40, 42] and the references therein. However, only a few papers [6–9] have dealt with supercritical sources, i.e., when $p > 3$ and $k > 2$.

In this paper we use the powerful theory of monotone operators and nonlinear semigroups (Kato's Theorem [5, 39]) to study system (1.1). Our strategy is similar to the one used by Bociu [6], and our proofs draw substantially from important ideas in [6, 8, 9] and in [12]. However, we were faced with the following technical issue: in the operator theoretic formulation of (1.1), although the operators induced by interior and boundary damping terms are individually maximal monotone from $H^1(\Omega)$ into $(H^1(\Omega))'$, it was crucial to verify that their sum is maximal monotone. Since neither of these two operators has the whole space $H^1(\Omega)$ as its domain, as the exponents m, r, and q of damping are arbitrarily large, then checking the domain condition (see Theorem 1.5 (p. 54) [5]) to assure maximal monotonicity of their sum becomes infeasible. In order to overcome this difficulty, we define a

new operator $\mathcal{S}$ which can represent the sum of interior and boundary damping. Indeed, the authors in [4] were able to generalize important results by Brézis [10] and provide a complete proof of the fact that $\mathcal{S}$ is a subdifferential of a convex functional, which immediately yields the fact that $\mathcal{S}$ is maximal monotone. Some details can be found in Subsection 2.1.

1.2. **Notation.** The following notation will be used throughout the paper:

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \ |u|_s = \|u\|_{L^s(\Gamma)}, \ \|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)},$$
$$(u,v)_\Omega = (u,v)_{L^2(\Omega)}, \ (u,v)_\Gamma = (u,v)_{L^2(\Gamma)}, \ (u,v)_{1,\Omega} = (u,v)_{H^1(\Omega)},$$
$$\tilde{m} = \frac{m+1}{m}, \ \tilde{r} = \frac{r+1}{r}, \ \tilde{q} = \frac{q+1}{q}.$$

As usual, we denote the standard duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ by $\langle \cdot, \cdot \rangle$. We also use the notation γu to denote the *trace* of u on Γ and we write $\frac{d}{dt}(\gamma u(t))$ as γu_t. In addition, the following Sobolev imbeddings will be used frequently, and sometimes without mention:

$$(1.3) \qquad \begin{cases} H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega), \ \text{for } \epsilon \in [0,1], \\ H^{1-\epsilon}(\Omega) \hookrightarrow H^{\frac{1}{2}-\epsilon}(\Gamma) \hookrightarrow L^{\frac{4}{1+2\epsilon}}(\Gamma), \ \text{for } \epsilon \in [0,\frac{1}{2}]. \end{cases}$$

We also remind the reader of the following interpolation inequality:

$$(1.4) \qquad \|u\|_{H^\theta(\Omega)}^2 \le \epsilon \|u\|_{1,\Omega}^2 + C(\epsilon,\theta) \|u\|_2^2,$$

for all $0 \le \theta < 1$ and $\epsilon > 0$. We finally note that $(\|\nabla u\|_2^2 + |\gamma u|_2^2)^{1/2}$ is an equivalent norm to the standard $H^1(\Omega)$ norm. This fact follows from a Poincaré-Wirtinger type of inequality:

$$\|u\|_2^2 \le C(\|\nabla u\|_2^2 + |\gamma u|_2^2) \ \text{ for all } u \in H^1(\Omega).$$

Thus, throughout the paper we put

$$(1.5) \qquad \|u\|_{1,\Omega}^2 = \|\nabla u\|_2^2 + |\gamma u|_2^2 \ \text{ and } \ (u,v)_{1,\Omega} = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma,$$

for $u, v \in H^1(\Omega)$.

1.3. **Main results.** In order to state our main result we begin by giving the definition of a weak solution to (1.1).

Definition 1.2. A pair of functions (u,v) is said to be a *weak solution* of (1.1) on $[0,T]$ if

- $u \in C([0,T];H^1(\Omega))$, $v \in C([0,T];H_0^1(\Omega))$, $u_t \in C([0,T];L^2(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$, $\gamma u_t \in L^{q+1}(\Gamma \times (0,T))$, $v_t \in C([0,T];L^2(\Omega)) \cap L^{r+1}(\Omega \times (0,T))$;
- $(u(0),v(0)) = (u_0,v_0) \in H^1(\Omega) \times H_0^1(\Omega)$, $(u_t(0),v_t(0)) = (u_1,v_1) \in L^2(\Omega) \times L^2(\Omega)$;

• for all $t \in [0, T]$, u and v verify the identities

$$(u_t(t), \phi(t))_\Omega - (u_t(0), \phi(0))_\Omega + \int_0^t [-(u_t(\tau), \phi_t(\tau))_\Omega + (u(\tau), \phi(\tau))_{1,\Omega}] d\tau$$

$$+ \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau) dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t(\tau))\gamma\phi(\tau) d\Gamma d\tau$$

(1.6)
$$= \int_0^t \int_\Omega f_1(u(\tau), v(\tau))\phi(\tau) dx d\tau + \int_0^t \int_\Gamma h(\gamma u(\tau))\gamma\phi(\tau) d\Gamma d\tau,$$

$$(v_t(t), \psi(t))_\Omega - (v_t(0), \psi(0))_\Omega + \int_0^t [-(v_t(\tau), \psi_t(\tau))_\Omega + (v(\tau), \psi(\tau))_{1,\Omega}] d\tau$$

(1.7)
$$+ \int_0^t \int_\Omega g_2(v_t(\tau))\psi(\tau) dx d\tau = \int_0^t \int_\Omega f_2(u(\tau), v(\tau))\psi(\tau) dx d\tau,$$

for all test functions satisfying:
$\phi \in C([0, T]; H^1(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$ such that $\gamma\phi \in L^{q+1}(\Gamma \times (0, T))$ with $\phi_t \in L^1([0, T]; L^2(\Omega))$ and $\psi \in C([0, T]; H_0^1(\Omega)) \cap L^{r+1}(\Omega \times (0, T))$ such that $\psi_t \in L^1([0, T]; L^2(\Omega))$.

Our first theorem establishes the existence of a local weak solution to (1.1). Specifically, we have the following result.

Theorem 1.3 (**Local weak solutions**). *Assume the validity of Assumption* 1.1. *Then there exists a local weak solution (u, v) to* (1.1) *defined on $[0, T_0]$ for some $T_0 > 0$ depending on the initial energy $E(0)$, where*

$$E(t) = \frac{1}{2}(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2).$$

In addition, the following energy identity holds for all $t \in [0, T_0]$:

$$E(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t] dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau$$

(1.8)
$$= E(0) + \int_0^t \int_\Omega [f_1(u, v)u_t + f_2(u, v)v_t] dx d\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau.$$

In order to state the next theorem, we need additional assumptions on the sources and the boundary damping.

Assumption 1.4.

(a) *For $p > 3$, there exists a function $F(u, v) \in C^3(\mathbb{R}^2)$ such that $f_1(u, v) = F_u(u, v)$, $f_2(u, v) = F_v(u, v)$ and $|D^\alpha F(u, v)| \leq C(|u|^{p-2} + |v|^{p-2} + 1)$, for all multi-indices $|\alpha| = 3$ and all $u, v \in \mathbb{R}$.*

(b) *For $k \geq 2$, $h \in C^2(\mathbb{R})$ such that $|h''(s)| \leq C(|s|^{k-2} + 1)$, for all $s \in \mathbb{R}$.*

(c) *For $k < 2$, there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$, for all $s_1, s_2 \in \mathbb{R}$.*

Theorem 1.5 (Uniqueness of weak solutions–Part 1). *In addition to Assumptions* 1.1 *and* 1.4, *we further assume that u_0, $v_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$ and $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. Then weak solutions of* (1.1) *are unique.*

Remark 1.6. The additional assumptions on the initial data in Theorem 1.5 are redundant if $p \leq 5$ and $k \leq 3$, due to the imbeddings (1.3). Also, it is often the

case that the interior sources f_1 and f_2 fail to satisfy Assumption 1.4(a), as in system (1.2) for the values $3 < p \leq 5$. To ensure uniqueness of weak solutions in such a case, we require the exponents m and r of the interior damping to be sufficiently large. More precisely, the following result resolves this issue.

Theorem 1.7 (Uniqueness of weak solutions–Part 2). *Under Assumption* 1.1 *and Assumption* 1.4*(b)(c), we additionally assume that* $u_0, v_0 \in L^{3(p-1)}(\Omega)$, $\gamma u_0 \in L^{2(k-1)}(\Gamma)$, *and* $m, r \geq 3p - 4$ *if* $p > 3$. *Then weak solutions of* (1.1) *are unique.*

Our next theorem states that weak solutions furnished by Theorem 1.3 are global solutions provided the exponents of damping are more dominant than the exponents of the corresponding sources.

Theorem 1.8 (Global weak solutions). *In addition to Assumption* 1.1, *further assume* $u_0, v_0 \in L^{p+1}(\Omega)$ *and* $\gamma u_0 \in L^{k+1}(\Gamma)$. *If* $p \leq \min\{m, r\}$ *and* $k \leq q$, *then the said solution* (u, v) *in Theorem* 1.3 *is a global weak solution and* T_0 *can be taken arbitrarily large.*

Remark 1.9. Recently, the authors [15, 16] obtained several blow up results for solutions with negative and nonnegative initial energy (the latter is for potential well solutions). Indeed, the results of [15,16] are for some cases when the conditions $p \leq \min\{m, r\}$ and $k \leq q$ are not fulfilled. More precisely, the results of [15, 16] show that every weak solution to (1.1) **blows up** in finite time provided either: (i) the interior and boundary sources are more dominant than their corresponding damping terms ($p > \max\{m, r\}$ and $k > q$) or (ii) the interior sources dominate both interior and boundary damping ($p > \max\{m, r, 2q - 1\}$).

Our final result states that the weak solution of (1.1) depends continuously on the initial data.

Theorem 1.10 (Continuous dependence on initial data). *Assume the validity of Assumptions* 1.1 *and* 1.4 *and an initial data* $U_0 = (u_0, v_0, u_1, v_1) \in X$, *where* X *is given by* $X := (H^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)) \times (H_0^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$, *such that* $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. *If* $U_0^n = (u_0^n, u_1^n, v_0^n, v_1^n)$ *is a sequence of initial data such that, as* $n \longrightarrow \infty$,

$$U_0^n \longrightarrow U_0 \ in \ X \ and \ \gamma u_0^n \longrightarrow \gamma u_0 \ in \ L^{2(k-1)}(\Gamma),$$

then the corresponding weak solutions (u^n, v^n) *and* (u, v) *of* (1.1) *satisfy*

$$(u^n, v^n, u_t^n, v_t^n) \longrightarrow (u, v, u_t, v_t) \ in \ C([0, T]; H), \ as \ n \longrightarrow \infty,$$

where $H := H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

Remark 1.11. If $p \leq 5$, then the spaces X and H in Theorem 1.10 are identical since $H^1(\Omega) \hookrightarrow L^6(\Omega)$. In addition, if $k \leq 3$, then the assumption $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{2(k-1)}(\Gamma)$ is redundant since $u_0^n \longrightarrow u_0$ in $H^1(\Omega)$ implies $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^4(\Gamma)$.

The paper is organized as follows. In Section 2, we provide a detailed proof of the local existence statement in Theorem 1.3. Section 3 is devoted to the derivation of the energy identity (1.8). In Section 4 we provide the proofs of the uniqueness statements of Theorems 1.5 and 1.7. Section 5 contains the proof of Theorem 1.8 and in Section 6 we provide the proof of Theorem 1.10.

2. LOCAL SOLUTIONS

This section is devoted to proving the existence statement in Theorem 1.3, which will be carried out in the following five subsections.

2.1. **Operator theoretic formulation.** Our first goal is to put problem (1.1) in an operator theoretic form. In order to do so, we introduce the Robin Laplacian Δ_R: $\mathcal{D}(\Delta_R) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$, where $\Delta_R = -\Delta u$ with its domain $\mathcal{D}(\Delta_R) = \{u \in H^2(\Omega) : \partial_\nu u + u = 0 \text{ on } \Gamma\}$. We note here that the Robin Laplacian can be extended to a continuous operator $\Delta_R : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ by

$$(2.1) \qquad \langle \Delta_R u, v \rangle = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma = (u, v)_{1,\Omega}$$

for all $u, v \in H^1(\Omega)$.

We also define the Robin map $R : H^s(\Gamma) \longrightarrow H^{s+\frac{3}{2}}(\Omega)$ as follows:

$$(2.2) \qquad q = Rp \Longleftrightarrow q \text{ is a weak solution for } \begin{cases} \Delta q = 0 & \text{in } \Omega, \\ \partial_\nu q + q = p & \text{on } \Gamma. \end{cases}$$

Hence, for $p \in L^2(\Gamma)$, we know from (2.2) that

$$(2.3) \qquad (Rp, \phi)_{1,\Omega} = (p, \gamma\phi)_\Gamma \quad \text{for all } \phi \in H^1(\Omega).$$

Combining (2.1) and (2.3) gives the following useful identity:

$$(2.4) \qquad \langle \Delta_R Rp, \phi \rangle = (Rp, \phi)_{1,\Omega} = (p, \gamma\phi)_\Gamma,$$

for all $p \in L^2(\Gamma)$ and $\phi \in H^1(\Omega)$.

By using the operators introduced above, we can put (1.1) in the following form:

$$(2.5) \qquad \begin{cases} u_{tt} + \Delta_R(u - Rh(\gamma u) + Rg(\gamma u_t)) + g_1(u_t) = f_1(u, v), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u, v), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H_0^1(\Omega), v_t(0) = v_1 \in L^2(\Omega). \end{cases}$$

It is important to point out here that in (2.5), we can show $\mathcal{S}_1 := \Delta_R Rg(\gamma u_t)$ and $\mathcal{S}_2 := g(u_t)$ are both maximal monotone from $H^1(\Omega)$ into $(H^1(\Omega))'$. However, in order to show that $\mathcal{S}_1 + \mathcal{S}_2$ is also maximal monotone, one needs to check the validity of domain condition: $(\text{int } \mathcal{D}(\mathcal{S}_1)) \cap \mathcal{D}(\mathcal{S}_2) \neq \emptyset$. The fact that the exponents of the interior and boundary damping, m and q, are allowed to be arbitrarily large makes it infeasible to verify the above domain condition.

In order to overcome this difficulty, we shall introduce a maximal monotone operator $\mathcal{S}$ representing the sum of interior and boundary damping. To do so, we first define the functional $J : H^1(\Omega) \longrightarrow [0, +\infty]$ by

$$(2.6) \qquad J(u) = \int_\Omega j_1(u)dx + \int_\Gamma j(\gamma u)d\Gamma,$$

where j_1 and $j : \mathbb{R} \longrightarrow [0, +\infty)$ are convex functions defined by

$$(2.7) \qquad j_1(s) = \int_0^s g_1(\tau)d\tau \text{ and } j(s) = \int_0^s g(\tau)d\tau.$$

Clearly, J is convex and lower semicontinuous. The subdifferential of J, $\partial J : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is defined by

$$(2.8) \quad \partial J(u) = \{u^* \in (H^1(\Omega))' : J(u) + \langle u^*, v - u \rangle \leq J(v) \text{ for all } v \in H^1(\Omega)\}.$$

The domain $\mathcal{D}(\partial J)$ represents the set of all functions $u \in H^1(\Omega)$ for which $\partial J(u)$ is nonempty.

By Theorem 2.2 in [4], we know that, for any $u \in \mathcal{D}(\partial J)$, $\partial J(u)$ is a singleton, and thus we may define the operator $\mathcal{S} : \mathcal{D}(\mathcal{S}) = \mathcal{D}(\partial J) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ such that

$$\tag{2.9} \partial J(u) = \{\mathcal{S}(u)\}.$$

It is well known that any subdifferential is maximal monotone, thus $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is a maximal monotone operator. Moreover, by Theorem 2.2 of [4], we also know that, for all $u \in \mathcal{D}(\mathcal{S})$, we have $g_1(u) \in L^1(\Omega)$, $g_1(u)u \in L^1(\Omega)$, $g(\gamma u) \in L^1(\Gamma)$ and $g(\gamma u)\gamma u \in L^1(\Gamma)$. In addition,

$$\tag{2.10} \langle \mathcal{S}(u), u \rangle = \int_\Omega g_1(u)u\, dx + \int_\Gamma g(\gamma u)\gamma u\, d\Gamma$$

and

$$\tag{2.11} \langle \mathcal{S}(u), v \rangle = \int_\Omega g_1(u)v\, dx + \int_\Gamma g(\gamma u)\gamma v\, d\Gamma \ \text{ for all } \ v \in C(\overline{\Omega}).$$

It follows that for all $u \in \mathcal{D}(S)$,

$$\tag{2.12} \langle \mathcal{S}(u), v \rangle = \int_\Omega g_1(u)v\, dx + \int_\Gamma g(\gamma u)\gamma v\, d\Gamma \ \text{ for all } \ v \in H^1(\Omega) \cap L^\infty(\Omega).$$

In fact, if $v \in H^1(\Omega) \cap L^\infty(\Omega)$, then there exists $v_n \in C(\overline{\Omega})$ such that $v_n \to v$ in $H^1(\Omega)$ and a.e. in Ω with $|v_n| \leq M$ in Ω for some $M > 0$. By (2.11) and the Lebesgue Dominated Convergence Theorem, we obtain (2.12).

By using the operator $\mathcal{S}$ we may rewrite (2.5) as

$$\tag{2.13} \begin{cases} u_{tt} + \Delta_R(u - Rh(\gamma u)) + \mathcal{S}(u_t) = f_1(u, v), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u, v), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H_0^1(\Omega), v_t(0) = v_1 \in L^2(\Omega). \end{cases}$$

It is important to note here that $\mathcal{S}(u_t)$ represents the sum of the interior damping $g(u_t)$ and the boundary damping $\Delta_R Rg(\gamma u_t)$. However, $\mathcal{D}(\mathcal{S})$ is not necessarily the same as the domain of the operator $\Delta_R Rg(\gamma \cdot) + g(\cdot) : H^1(\Omega) \longrightarrow (H^1(\Omega))'$. Therefore, systems (2.5) and (2.13) are not exactly equivalent. Nonetheless, we shall see that if (u, v) is a strong solution for (2.13), then (u, v) must be a weak solution for (1.1) in the sense of Definition 1.2. So, instead of studying (1.1) directly, we first show that system (2.13) has a unique strong solution.

Let $H = H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and define the nonlinear operator

$$\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \longrightarrow H$$

by

$$\tag{2.14} \mathscr{A} \begin{bmatrix} u \\ v \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ \Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u, v) \\ -\Delta v + g_2(z) - f_2(u, v) \end{bmatrix}^{tr},$$

where

$$\mathcal{D}(\mathscr{A}) = \Big\{ (u, v, y, z) \in \Big(H^1(\Omega) \times H_0^1(\Omega) \Big)^2 :$$

$$\Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u, v) \in L^2(\Omega), \ y \in \mathcal{D}(\mathcal{S}),$$

$$- \Delta v + g_2(z) - f_2(u, v) \in L^2(\Omega), \ g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega) \Big\}.$$

Put $U = (u, v, u_t, v_t)$. Then the system (2.13) is equivalent to

(2.15) $$U_t + \mathscr{A}U = 0 \ \text{ with } \ U(0) = (u_0, v_0, u_1, v_1) \in H.$$

2.2. **Globally Lipschitz sources.** First, we deal with the case where the boundary damping is assumed strongly monotone and the sources are globally Lipschitz. In this case, we have the following lemma.

Lemma 2.1. *Assume that,*

- *g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0)$ $= g_2(0) = g(0) = 0$. Moreover, the following strong monotonicity condition is imposed on g:*
  there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g |s_1 - s_2|^2$.
- *f_1, $f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ are globally Lipschitz.*
- *$h \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is globally Lipschitz.*

Then, system (2.13) has a unique global strong solution $U \in W^{1,\infty}(0, T; H)$ for arbitrary $T > 0$, provided the datum $U_0 \in \mathcal{D}(\mathscr{A})$.

Proof. In order to prove Lemma 2.1 it suffices to show that the operator $\mathscr{A} + \omega I$ is m-accretive for some positive ω. We say an operator $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \longrightarrow H$ is *accretive* if $(\mathscr{A}x_1 - \mathscr{A}x_2, x_1 - x_2)_H \geq 0$, for all $x_1, x_2 \in \mathcal{D}(\mathscr{A})$, and it is m-*accretive* if, in addition, $\mathscr{A} + I$ maps $\mathcal{D}(\mathscr{A})$ onto H. In fact, by Kato's Theorem (see [39] for instance), if $\mathscr{A} + \omega I$ is m-accretive for some positive ω, then for each $U_0 \in \mathcal{D}(\mathscr{A})$ there is a unique strong solution U of (2.15), i.e., $U \in W^{1,\infty}(0, T; H)$ such that $U(0) = U_0$, $U(t) \in \mathcal{D}(\mathscr{A})$ for all $t \in [0, T]$, and equation (2.15) is satisfied a.e. $[0, T]$, where $T > 0$ is arbitrary.

Step 1: *Proof for $\mathscr{A} + \omega I$ is accretive for some positive ω.* Let $U = [u, v, y, z]$, $\hat{U} = [\hat{u}, \hat{v}, \hat{y}, \hat{z}] \in \mathcal{D}(\mathscr{A})$. We aim to find $\omega > 0$ such that

$$((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H \geq 0.$$

By straightforward calculations, we obtain

(2.16)

$$((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H = (\mathscr{A}(U) - \mathscr{A}(\hat{U}), U - \hat{U})_H + \omega |U - \hat{U}|_H^2$$

$$= -(y - \hat{y}, u - \hat{u})_{1,\Omega} - (z - \hat{z}, v - \hat{v})_{1,\Omega} + \langle \Delta_R(u - \hat{u}), y - \hat{y} \rangle$$

$$- \langle \Delta_R R(h(\gamma u) - h(\gamma \hat{u})), y - \hat{y} \rangle + \langle \mathcal{S}(y) - \mathcal{S}(\hat{y}), y - \hat{y} \rangle$$

$$- (f_1(u, v) - f_1(\hat{u}, \hat{v}), y - \hat{y})_\Omega - \langle \Delta(v - \hat{v}), z - \hat{z} \rangle$$

$$+ \langle g_2(z) - g_2(\hat{z}), z - \hat{z} \rangle - (f_2(u, v) - f_2(\hat{u}, \hat{v}), z - \hat{z})_\Omega$$

$$+ \omega(\|u - \hat{u}\|_{1,\Omega}^2 + \|v - \hat{v}\|_{1,\Omega}^2 + \|y - \hat{y}\|_2^2 + \|z - \hat{z}\|_2^2).$$

Notice

(2.17) $$- \langle \Delta(v - \hat{v}), z - \hat{z} \rangle = (\nabla(v - \hat{v}), \nabla(z - \hat{z}))_\Omega = (v - \hat{v}, z - \hat{z})_{1,\Omega}.$$

Moreover, since $g_2(y) - g_2(\hat{y}) \in H^{-1}(\Omega) \cap L^1(\Omega)$ and $z - \hat{z} \in H_0^1(\Omega)$ satisfying $(g_2(z(x)) - g_2(\hat{z}(x)))(z(x) - \hat{z}(x)) \geq 0$, for all $x \in \Omega$, then by Lemma 2.2 (p. 89) in [5], we have $(g_2(z) - g_2(\hat{z}))(z - \hat{z}) \in L^1(\Omega)$ and

$$(2.18) \qquad \langle g_2(z) - g_2(\hat{z}), z - \hat{z} \rangle = \int_\Omega (g_2(z) - g_2(\hat{z}))(z - \hat{z})dx \geq 0.$$

Now we show

$$\begin{aligned}
&\langle \mathcal{S}(y) - \mathcal{S}(\hat{y}), y - \hat{y} \rangle \\
(2.19) \qquad &\geq \int_\Omega (g_1(y) - g_1(y))(y - \hat{y})dx + \int_\Gamma (g(\gamma y) - g(\gamma \hat{y}))(\gamma y - \gamma \hat{y})d\Gamma.
\end{aligned}$$

Since $y - \hat{y} \in H^1(\Omega)$, if we set

$$(2.20) \qquad w_n = \begin{cases} n & \text{if } y - \hat{y} \geq n, \\ y - \hat{y} & \text{if } |y - \hat{y}| \leq n, \\ -n & \text{if } y - \hat{y} \leq -n, \end{cases}$$

then $w_n \in H^1(\Omega) \cap L^\infty(\Omega)$. So by (2.12) one has

$$(2.21) \quad \langle \mathcal{S}(y) - \mathcal{S}(\hat{y}), w_n \rangle = \int_\Omega (g_1(y) - g_1(\hat{y}))w_n dx + \int_\Gamma (g(\gamma y) - g(\gamma \hat{y}))\gamma w_n d\Gamma.$$

Moreover, by (2.20) we know w_n and $y - \hat{y}$ have the same sign; then since g_1 is monotone increasing, one has $(g_1(y) - g_1(\hat{y}))w_n \geq 0$ a.e. in Ω. Therefore, by Fatou's Lemma, we obtain

$$(2.22) \qquad \liminf_{n \to \infty} \int_\Omega (g_1(y) - g_1(\hat{y}))w_n dx \geq \int_\Omega (g_1(y) - g_1(\hat{y}))(y - \hat{y})dx.$$

Likewise, we have

$$(2.23) \qquad \liminf_{n \to \infty} \int_\Gamma (g(\gamma y) - g(\gamma \hat{y}))\gamma w_n d\Gamma \geq \int_\Gamma (g(\gamma y) - g(\gamma \hat{y}))(\gamma y - \gamma \hat{y})d\Gamma.$$

Since $w_n \to y - \hat{y}$ in $H^1(\Omega)$, by taking the lower limit on both sides of (2.21) and using (2.22)-(2.23), we conclude that the inequality (2.19) holds.

By using (2.1), (2.4), (2.17), (2.18) and (2.19), we obtain from (2.16) that

$$\begin{aligned}
&((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H \\
&\geq (g(\gamma y) - g(\gamma \hat{y}), \gamma y - \gamma \hat{y})_\Gamma - (h(\gamma u) - h(\gamma \hat{u}), \gamma y - \gamma \hat{y})_\Gamma \\
&\quad - (f_1(u,v) - f_1(\hat{u},\hat{v}), y - \hat{y})_\Omega - (f_2(u,v) - f_2(\hat{u},\hat{v}), z - \hat{z})_\Omega \\
(2.24) \qquad &\quad + \omega(\|u - \hat{u}\|_{1,\Omega}^2 + \|v - \hat{v}\|_{1,\Omega}^2 + \|y - \hat{y}\|_2^2 + \|z - \hat{z}\|_2^2).
\end{aligned}$$

Let $V = H^1(\Omega) \times H_0^1(\Omega)$ and recall the assumption that f_1, f_2 and h are globally Lipschitz continuous with Lipschitz constant L_{f_1}, L_{f_2}, and L_h, respectively. Let $L = \max\{L_{f_1}, L_{f_2}, L_h\}$. Therefore, by employing the strong monotonicity condition

on g and Young's inequality, we have

(2.25)
$$
(g(\gamma y) - g(\gamma \hat{y}), \gamma y - \gamma \hat{y})_\Gamma - (h(\gamma u) - h(\gamma \hat{u}), \gamma y - \gamma \hat{y})_\Gamma
$$
$$
- (f_1(u,v) - f_1(\hat{u}, \hat{v}), y - \hat{y})_\Omega - (f_2(u,v) - f_2(\hat{u}, \hat{v}), z - \hat{z})_\Omega
$$
$$
\geq m_g |\gamma y - \gamma \hat{y}|_2^2 - L\, \|u - \hat{u}\|_{1,\Omega}\, |\gamma y - \gamma \hat{y}|_2 - L\, \|(u - \hat{u}, v - \hat{v})\|_V\, \|y - \hat{y}\|_2
$$
$$
- L\, \|(u - \hat{u}, v - \hat{v})\|_V\, \|z - \hat{z}\|_2
$$
$$
\geq m_g |\gamma y - \gamma \hat{y}|_2^2 - \frac{L^2}{4\epsilon} \|u - \hat{u}\|_{1,\Omega}^2 - \epsilon|\gamma y - \gamma \hat{y}|_2^2 - \frac{L}{2}(\|u - \hat{u}\|_{1,\Omega}^2 + \|v - \hat{v}\|_{1,\Omega}^2)
$$
$$
- \frac{L}{2}\|y - \hat{y}\|_2^2 - \frac{L}{2}(\|u - \hat{u}\|_{1,\Omega}^2 + \|v - \hat{v}\|_{1,\Omega}^2) - \frac{L}{2}\|z - \hat{z}\|_2^2.
$$

Combining (2.24) and (2.25) leads to

$$
((\mathscr{A} + \omega I)U - (\mathscr{A} + \omega I)\hat{U}, U - \hat{U})_H
$$
$$
\geq (m_g - \epsilon)|\gamma y - \gamma \hat{y}|_2^2 + (\omega - \frac{L^2}{4\epsilon} - L)\|u - \hat{u}\|_{1,\Omega}^2
$$
$$
+ (\omega - L)\|v - \hat{v}\|_{1,\Omega}^2 + (\omega - \frac{L}{2})\|y - \hat{y}\|_2^2 + (\omega - \frac{L}{2})\|z - \hat{z}\|_2^2.
$$

Therefore, by choosing $\epsilon < m_g$ and $\omega > \frac{L^2}{4\epsilon} + L$, then $\mathscr{A} + \omega I$ is accretive.

Step 2: Proof for $\mathscr{A} + \lambda I$ is m-accretive, for some $\lambda > 0$. To this end, it suffices to show that the range of $\mathscr{A} + \lambda I$ is all of H, for some $\lambda > 0$.

Let $(a, b, c, d) \in H$. We have to show that there exists $(u, v, y, z) \in \mathcal{D}(\mathscr{A})$ such that $(\mathscr{A} + \lambda I)(u, v, y, z) = (a, b, c, d)$, for some $\lambda > 0$, i.e.,

(2.26)
$$
\begin{cases}
-y + \lambda u = a, \\
-z + \lambda v = b, \\
\Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u,v) + \lambda y = c, \\
-\Delta v + g_2(z) - f_2(u,v) + \lambda z = d.
\end{cases}
$$

Note that (2.26) is equivalent to

(2.27)
$$
\begin{cases}
\frac{1}{\lambda}\Delta_R(y) - \Delta_R Rh\left(\gamma \frac{a+y}{\lambda}\right) + \mathcal{S}(y) - f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda y = c - \frac{1}{\lambda}\Delta_R(a), \\
-\frac{1}{\lambda}\Delta z + g_2(z) - f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda z = d + \frac{1}{\lambda}\Delta b.
\end{cases}
$$

Recall that $V = H^1(\Omega) \times H_0^1(\Omega)$ and notice that the right hand side of (2.27) belongs to V'. Thus, we define the operator $\mathscr{B} : \mathcal{D}(\mathscr{B}) \subset V \longrightarrow V'$ by

$$
\mathscr{B} \left[\begin{array}{c} y \\ z \end{array} \right]^{tr} = \left[\begin{array}{c} \frac{1}{\lambda}\Delta_R(y) - \Delta_R Rh\left(\gamma\frac{a+y}{\lambda}\right) + \mathcal{S}(y) - f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda y \\ -\frac{1}{\lambda}\Delta z + g_2(z) - f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda z \end{array} \right]^{tr},
$$

where $\mathcal{D}(\mathscr{B}) = \{(y,z) \in V : y \in \mathcal{D}(\mathcal{S}), g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$. Therefore, the issue reduces to proving that $\mathscr{B} : \mathcal{D}(\mathscr{B}) \subset V \longrightarrow V'$ is surjective. By Corollary 1.2 (p. 45) in [5], it is enough to show that $\mathscr{B}$ is maximal monotone and coercive.

We split $\mathscr{B}$ as two operators:

$$\mathscr{B}_1 \begin{bmatrix} y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} \frac{1}{\lambda}\Delta_R(y) - \Delta_R Rh\left(\gamma\frac{a+y}{\lambda}\right) - f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda y \\ -\frac{1}{\lambda}\Delta z - f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) + \lambda z \end{bmatrix}^{tr}$$

and

$$\mathscr{B}_2 \begin{bmatrix} y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} \mathcal{S}(y) \\ g_2(z) \end{bmatrix}^{tr}.$$

$\mathscr{B}_1$ is maximal monotone and coercive. First we note $\mathcal{D}(\mathscr{B}_1) = V$. To see that $\mathscr{B}_1 : V \longrightarrow V'$ is monotone, we let $Y = (y, z) \in V$ and $\hat{Y} = (\hat{y}, \hat{z}) \in V$. By straightforward calculations, we obtain

$$\begin{aligned}
&\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \\
&= \frac{1}{\lambda}\langle \Delta_R(y - \hat{y}), y - \hat{y} \rangle - \left\langle \Delta_R R\left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right)\right), y - \hat{y} \right\rangle \\
&\quad - \left(f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_1\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), y - \hat{y}\right)_\Omega \\
&\quad + \lambda\|y - \hat{y}\|_2^2 - \frac{1}{\lambda}\langle \Delta(z - \hat{z}), z - \hat{z} \rangle \\
&\quad - \left(f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_2\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), z - \hat{z}\right)_\Omega + \lambda\|z - \hat{z}\|_2^2.
\end{aligned}$$

By (2.1) and (2.4) we have

$$\begin{aligned}
&\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \\
&= \frac{1}{\lambda}(y - \hat{y}, y - \hat{y})_{1,\Omega} - \left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right), \gamma y - \gamma\hat{y}\right)_\Gamma \\
&\quad - \left(f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_1\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), y - \hat{y}\right)_\Omega \\
&\quad + \lambda\|y - \hat{y}\|_2^2 + \frac{1}{\lambda}(z - \hat{z}, z - \hat{z})_{1,\Omega} \\
&\quad - \left(f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_2\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), z - \hat{z}\right)_\Omega + \lambda\|z - \hat{z}\|_2^2.
\end{aligned}$$

Since f_1, f_2, h are Lipschitz continuous with Lipschitz constant L,

$$\begin{aligned}
\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle &\geq \frac{1}{\lambda}\|y - \hat{y}\|_{1,\Omega}^2 - \frac{L}{\lambda}\|y - \hat{y}\|_{1,\Omega}|\gamma y - \gamma\hat{y}|_2 \\
&\quad - \frac{L}{\lambda}\|(y - \hat{y}, z - \hat{z})\|_V\|y - \hat{y}\|_2 + \lambda\|y - \hat{y}\|_2^2 + \frac{1}{\lambda}\|z - \hat{z}\|_{1,\Omega}^2 \\
&\quad - \frac{L}{\lambda}\|(y - \hat{y}, z - \hat{z})\|_V\|z - \hat{z}\|_2 + \lambda\|z - \hat{z}\|_2^2.
\end{aligned}$$

Applying Young's inequality yields

$$\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \geq \frac{1}{\lambda} \|y - \hat{y}\|_{1,\Omega}^2 - \frac{L^2}{4\eta\lambda} \|y - \hat{y}\|_{1,\Omega}^2 - \frac{\eta}{\lambda} |\gamma y - \gamma \hat{y}|_2^2$$

$$- \frac{L^2}{4\eta\lambda} (\|y - \hat{y}\|_{1,\Omega}^2 + \|z - \hat{z}\|_{1,\Omega}^2) - \frac{\eta}{\lambda} \|y - \hat{y}\|_2^2 + \lambda \|y - \hat{y}\|_2^2 + \frac{1}{\lambda} \|z - \hat{z}\|_{1,\Omega}^2$$

$$- \frac{L^2}{4\eta\lambda} (\|y - \hat{y}\|_{1,\Omega}^2 + \|z - \hat{z}\|_{1,\Omega}^2) - \frac{\eta}{\lambda} \|z - \hat{z}\|_2^2 + \lambda \|z - \hat{z}\|_2^2$$

$$\geq \left(\frac{1}{\lambda} - \frac{3L^2}{4\eta\lambda} \right) \|y - \hat{y}\|_{1,\Omega}^2 - \frac{\eta}{\lambda} |\gamma y - \gamma \hat{y}|_2^2$$

$$+ \left(\frac{1}{\lambda} - \frac{2L^2}{4\eta\lambda} \right) \|z - \hat{z}\|_{1,\Omega}^2 + \left(\lambda - \frac{\eta}{\lambda} \right) (\|y - \hat{y}\|_2^2 + \|z - \hat{z}\|_2^2).$$

By using the imbedding $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^2(\Gamma)$ and the interpolation inequality (1.4), we obtain

$$|\gamma u|_2^2 \leq C \|u\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \delta \|u\|_{1,\Omega}^2 + C_\delta \|u\|_2^2,$$

for all $u \in H^1(\Omega)$, where $\delta > 0$. It follows that

$$|\gamma y - \gamma \hat{y}|_2^2 \leq \delta \|y - \hat{y}\|_{1,\Omega}^2 + C_\delta \|y - \hat{y}\|_2^2.$$

Thus,

$$\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \geq \left(\frac{1}{2\lambda} - \frac{3L^2}{4\eta\lambda} - \frac{\eta\delta}{\lambda} \right) \|y - \hat{y}\|_{1,\Omega}^2$$

$$+ (\lambda - \frac{\eta + \eta C_\delta}{\lambda}) \|y - \hat{y}\|_2^2 + (\lambda - \frac{\eta}{\lambda}) \|z - \hat{z}\|_2^2 + \left(\frac{1}{2\lambda} - \frac{2L^2}{4\eta\lambda} \right) \|z - \hat{z}\|_{1,\Omega}^2$$

$$+ \frac{1}{2\lambda} (\|y - \hat{y}\|_{1,\Omega}^2 + \|z - \hat{z}\|_{1,\Omega}^2).$$

Note that the sign of

$$\frac{1}{2\lambda} - \frac{3L^2}{4\eta\lambda} - \frac{\eta\delta}{\lambda} = \frac{2 - 3L^2/\eta - 4\eta\delta}{4\lambda}$$

does not depend on the value of λ. So, we let $\eta > 3L^2$ and choose $\delta > 0$ sufficiently small so that $4\eta\delta < 1$. In addition, we select λ sufficiently large such that $\lambda^2 > \eta + \eta C_\delta$. Therefore,

$$\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \geq \frac{1}{2\lambda} (\|y - \hat{y}\|_{1,\Omega}^2 + \|z - \hat{z}\|_{1,\Omega}^2) = \frac{1}{2\lambda} \|Y - \hat{Y}\|_V^2,$$

proving that $\mathscr{B}_1$ is strongly monotone. It is easy to see that strong monotonicity implies coercivity of $\mathscr{B}_1$.

Next, we show that $\mathscr{B}_1$ is continuous. Clearly, $\Delta_R : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ and $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ are continuous. Moreover, if we set

$$\tilde{f}_j(y, z) := f_j \left(\frac{a + y}{\lambda}, \frac{b + z}{\lambda} \right), \quad j = 1, 2,$$

then, since $f_1, f_2 : V \longrightarrow L^2(\Omega)$ are globally Lipschitz, it is clear that the mappings $\tilde{f}_1 : V \longrightarrow (H^1(\Omega))'$ and $\tilde{f}_2 : V \longrightarrow H^{-1}(\Omega)$ are also Lipschitz continuous.

To see that the mapping

$$\tilde{h}(y) := \Delta_R R h \left(\gamma \frac{a + y}{\lambda} \right)$$

is Lipschitz continuous from $H^1(\Omega)$ into $(H^1(\Omega))'$, we use (2.4) and the assumption that $h \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is globally Lipschitz continuous. Indeed,

$$\left\| \tilde{h}(y) - \tilde{h}(\hat{y}) \right\|_{(H^1(\Omega))'} = \sup_{\|\varphi\|_{1,\Omega}=1} \left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right), \gamma\varphi \right)_\Gamma$$

$$\leq C \left| h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right) \right|_2 \leq \frac{CL}{\lambda}\|y - \hat{y}\|_{1,\Omega}.$$

It follows that $\mathscr{B}_1 : V \longrightarrow V'$ is continuous, and along with the monotonicity of $\mathscr{B}_1$, we conclude that $\mathscr{B}_1$ is maximal monotone.

$\mathscr{B}_2$ **is maximal monotone.** First we note that $\mathcal{D}(\mathscr{B}_2) = \mathcal{D}(\mathscr{B}) = \{(y, z) \in V : y \in \mathcal{D}(\mathcal{S}), g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$. Remember in Subsection 2.1 we have already shown that $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is maximal monotone. In order to study the operator $g_2(z)$, we define the functional $J_2 : H_0^1(\Omega) \longrightarrow [0, \infty]$ by

$$J_2(z) = \int_\Omega j_2(z(x))dx,$$

where $j_2 : \mathbb{R} \longrightarrow [0, +\infty)$ is a convex function defined by

$$j_2(s) = \int_0^s g_2(\tau)d\tau.$$

Clearly J_2 is proper, convex and lower semicontinuous. Moreover, by Corollary 2.3 in [4] we know that $\partial J_2 : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ satisfies

(2.28) $\partial J_2(z) = \{\mu \in H^{-1}(\Omega) \cap L^1(\Omega) : \mu = g_2(z) \text{ a.e. in } \Omega\}.$

That is to say, $\mathcal{D}(\partial J_2) = \{z \in H_0^1(\Omega) : g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$, and for all $z \in \mathcal{D}(\partial J_2)$, $\partial J_2(z)$ is a singleton such that $\partial J_2(z) = \{g_2(z)\}$. Since any subdifferential is maximal monotone, we obtain the maximal monotonicity of the operator $g_2(\cdot) : \mathcal{D}(\partial J_2) \subset H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$. Hence, by Proposition 7.1 in the Appendix, it follows that $\mathscr{B}_2 : \mathcal{D}(\mathscr{B}_2) \subset V \longrightarrow V'$ is maximal monotone. Now, since $\mathscr{B}_1$ and $\mathscr{B}_2$ are both maximal monotone and $\mathcal{D}(\mathscr{B}_1) = V$, we conclude that $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is maximal monotone.

Finally, since $\mathscr{B}_2$ is monotone and $\mathscr{B}_2 0 = 0$, it follows that $\langle \mathscr{B}_2 Y, Y \rangle \geq 0$ for all $Y \in \mathcal{D}(\mathcal{S})$, and along with the fact $\mathscr{B}_1$ is coercive, we obtain that $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is coercive as well. Then, the surjectivity of $\mathscr{B}$ follows immediately by Corollary 1.2 (p. 45) in [5]. Thus, we have proved the existence of (y, z) in $\mathcal{D}(\mathscr{B}) \subset V = H^1(\Omega) \times H_0^1(\Omega)$ such that (y, z) satisfies (2.27). So by (2.26), $(u, v) = (\frac{y+a}{\lambda}, \frac{z+b}{\lambda}) \in H^1(\Omega) \times H_0^1(\Omega)$. In addition, one can easily see that $(u, v, y, z) \in \mathcal{D}(\mathscr{A})$. Indeed, we have $\Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u, v) = -\lambda y + c \in L^2(\Omega)$ and $-\Delta v + g_2(z) - f_2(u, v) = -\lambda z + d \in L^2(\Omega)$. Thus, the proof of maximal accretivity is completed, and so is the proof of Lemma 2.1. $\square$

2.3. **Locally Lipschitz sources.** In this subsection, we loosen the restrictions on sources and allow f_1, f_2 and h to be locally Lipschitz continuous.

Lemma 2.2. *For $m, r, q \geq 1$, we assume that:*
- *g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0) = g_2(0) = g(0) = 0$. In addition, the following growth conditions hold: there exist $\alpha > 0$ such that $g_1(s)s \geq \alpha|s|^{m+1}$, $g_2(s)s \geq \alpha|s|^{r+1}$ and $g(s)s \geq \alpha|s|^{q+1}$ for $|s| \geq 1$. Moreover, there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$.*

- f_1, $f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ *are locally Lipschitz continuous.*
- $h \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ *is locally Lipschitz continuous.*

Then, system (2.13) *has a unique local strong solution* $U \in W^{1,\infty}(0, T_0; H)$ *for some* $T_0 > 0$, *provided the initial datum* $U_0 \in \mathcal{D}(\mathscr{A})$.

Proof. As in [9,12], we use standard truncation of the sources. Recall $V = H^1(\Omega) \times H_0^1(\Omega)$ and define

$$f_1^K(u,v) = \begin{cases} f_1(u,v) & \text{if } \|(u,v)\|_V \leq K, \\ f_1\left(\frac{Ku}{\|(u,v)\|_V}, \frac{Kv}{\|(u,v)\|_V}\right) & \text{if } \|(u,v)\|_V > K, \end{cases}$$

$$f_2^K(u,v) = \begin{cases} f_2(u,v) & \text{if } \|(u,v)\|_V \leq K, \\ f_2\left(\frac{Ku}{\|(u,v)\|_V}, \frac{Kv}{\|(u,v)\|_V}\right) & \text{if } \|(u,v)\|_V > K, \end{cases}$$

$$h^K(u) = \begin{cases} h(\gamma u) & \text{if } \|u\|_{1,\Omega} \leq K, \\ h\left(\gamma \frac{Ku}{\|u\|_{1,\Omega}}\right) & \text{if } \|u\|_{1,\Omega} > K, \end{cases}$$

where K is a positive constant such that $K^2 \geq 4E(0) + 1$, where the energy $E(t)$ is given by $E(t) = \frac{1}{2}\left(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right)$.

With the truncated sources above, we consider the following K problem:

$$(K) \begin{cases} u_{tt} + \Delta_R(u - Rh^K(u)) + \mathcal{S}(u_t) = f_1^K(u,v) & \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + g_2(v_t) = f_2^K(u,v) & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \in H^1(\Omega), u_t(x,0) = u_1(x) \in H^1(\Omega), \\ v(x,0) = v_0(x) \in H_0^1(\Omega), v_t(x,0) = v_1(x) \in H_0^1(\Omega). \end{cases}$$

We note here that for each such K, the operators f_1^K, $f_2^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ and $h^K : H^1(\Omega) \longrightarrow L^2(\Gamma)$ are globally Lipschitz continuous (see [12]). Therefore, by Lemma 2.1, the (K) problem has a unique global strong solution $U_K \in W^{1,\infty}(0,T; H)$ for any $T > 0$, provided the initial datum $U_0 \in \mathcal{D}(\mathscr{A})$.

In what follows, we shall express $(u_K(t), v_K(t))$ as $(u(t), v(t))$. Since $u_t \in \mathcal{D}(\mathcal{S}) \subset H^1(\Omega)$ and $v_t \in H_0^1(\Omega)$ such that $g(v_t) \in H^{-1}(\Omega) \cap L^1(\Omega)$, then by (2.10) and Lemma 2.2 (p. 89) in [5], we may use the multipliers u_t and v_t on the K problem and obtain the following energy identity:

$$E(t) + \int_0^t \int_\Omega \left(g_1(u_t)u_t + g_2(v_t)v_t\right)dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau$$

$$(2.29) \quad = E(0) + \int_0^t \int_\Omega (f_1^K(u,v)u_t + f_2^K(u,v)v_t)dxd\tau + \int_0^t \int_\Gamma h^K(u)\gamma u_t d\Gamma d\tau.$$

In addition, since m, r, $q \geq 1$, we know $\tilde{m} = \frac{m+1}{m}$, $\tilde{r} = \frac{r+1}{r}$, $\tilde{q} = \frac{q+1}{q} \leq 2$. Hence, by our assumptions on the sources, it follows that $f_1: H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, $f_2: H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$, and $h \circ \gamma: H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ are all locally Lipschitz with Lipschitz constant $L_{f_1}(K)$, $L_{f_2}(K)$, $L_h(K)$, respectively, on the ball $\{(u,v) \in V : \|(u,v)\|_V \leq K\}$. Put

$$L_K = \max\{L_{f_1}(K), L_{f_2}(K), L_h(K)\}.$$

By using similar calculations as in [12], we deduce $f_1^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, $f_2^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$ and $h^K : H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ are globally Lipschitz with Lipschitz constant L_K.

We now estimate the terms due to the sources in the energy identity (2.29). By using Hölder's and Young's inequalities, we have

(2.30)
$$\int_0^t \int_\Omega f_1^K(u,v)u_t dxd\tau \le \int_0^t \left\| f_1^K(u,v) \right\|_{\tilde{m}} \left\| u_t \right\|_{m+1} d\tau$$
$$\le \epsilon \int_0^t \left\| u_t \right\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left\| f_1^K(u,v) \right\|_{\tilde{m}}^{\tilde{m}} d\tau$$
$$\le \epsilon \int_0^t \left\| u_t \right\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\left\| f_1^K(u,v) - f_1^K(0,0) \right\|_{\tilde{m}}^{\tilde{m}} d + \left\| f_1^K(0,0) \right\|_{\tilde{m}}^{\tilde{m}} \right) d\tau$$
$$\le \epsilon \int_0^t \left\| u_t \right\|_{m+1}^{m+1} d\tau + C_\epsilon L_K^{\tilde{m}} \int_0^t (\left\| u \right\|_{1,\Omega}^{\tilde{m}} + \left\| v \right\|_{1,\Omega}^{\tilde{m}}) d\tau + C_\epsilon t |f_1(0,0)|^{\tilde{m}} |\Omega|.$$

Likewise, we deduce

(2.31)
$$\int_0^t \int_\Omega f_2^K(u,v)v_t dxd\tau$$
$$\le \epsilon \int_0^t \left\| v_t \right\|_{r+1}^{r+1} d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t (\left\| u \right\|_{1,\Omega}^{\tilde{r}} + \left\| v \right\|_{1,\Omega}^{\tilde{r}}) d\tau + C_\epsilon t |f_2(0,0)|^{\tilde{r}} |\Omega|$$

and

(2.32)
$$\int_0^t \int_\Gamma h^K(u)\gamma u_t d\Gamma d\tau \le \epsilon \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \left\| u \right\|_{1,\Omega}^{\tilde{q}} d\tau + C_\epsilon t |h(0)|^{\tilde{q}} |\Gamma|.$$

By the assumptions on damping, it follows that

(2.33) $g_1(s)s \ge \alpha(|s|^{m+1} - 1), \quad g_2(s)s \ge \alpha(|s|^{r+1} - 1), \quad g(s)s \ge \alpha(|s|^{q+1} - 1)$

for all $s \in \mathbb{R}$. Therefore,

(2.34)
$$\begin{cases} \int_0^t \int_\Omega g_1(u_t)u_t dxd\tau \ge \alpha \int_0^t \left\| u_t \right\|_{m+1}^{m+1} d\tau - \alpha t |\Omega|, \\ \int_0^t \int_\Omega g_2(v_t)v_t dxd\tau \ge \alpha \int_0^t \left\| v_t \right\|_{r+1}^{r+1} d\tau - \alpha t |\Omega|, \\ \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \ge \alpha \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau - \alpha t |\Gamma|. \end{cases}$$

By combining (2.30)-(2.34) in the energy identity (2.29), one has

$$E(t) + \alpha \int_0^t (\left\| u_t \right\|_{m+1}^{m+1} + \left\| v_t \right\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau - \alpha t (2|\Omega| + |\Gamma|)$$
$$\le E(0) + \epsilon \int_0^t (\left\| u_t \right\|_{m+1}^{m+1} + \left\| v_t \right\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau$$
$$+ C_\epsilon L_K^{\tilde{m}} \int_0^t (\left\| u \right\|_{1,\Omega}^{\tilde{m}} + \left\| v \right\|_{1,\Omega}^{\tilde{m}}) d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t (\left\| u \right\|_{1,\Omega}^{\tilde{r}} + \left\| v \right\|_{1,\Omega}^{\tilde{r}}) d\tau$$
(2.35) $$+ C_\epsilon L_K^{\tilde{q}} \int_0^t \left\| u \right\|_{1,\Omega}^{\tilde{q}} d\tau + C_\epsilon t (|f_1(0,0)|^{\tilde{m}} |\Omega| + |f_2(0,0)|^{\tilde{r}} |\Omega| + |h(0)|^{\tilde{q}} |\Gamma|).$$

If $\epsilon \leq \alpha$, then (2.35) implies

$$
\begin{aligned}
E(t) \leq & E(0) + C_\epsilon L_K^{\tilde{m}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau \\
& + C_\epsilon L_K^{\tilde{r}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u\|_{1,\Omega}^{\tilde{q}} d\tau
\end{aligned}
$$

(2.36) $\qquad + C_\epsilon t(|f_1(0,0)|^{\tilde{m}}|\Omega| + |f_2(0,0)|^{\tilde{r}}|\Omega| + |h(0)|^{\tilde{q}}|\Gamma|) + \alpha t(2|\Omega| + |\Gamma|).$

Since $\tilde{m}, \tilde{r}, \tilde{q} \leq 2$, then by Young's inequality,

$$
\int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau \leq \int_0^t (\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 + \tilde{C}) d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C} t,
$$

$$
\int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C} t,
$$

$$
\int_0^t \|u\|_{1,\Omega}^{\tilde{q}} d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C} t,
$$

where $\tilde{C}$ is a positive constant that depends on m, r and q. Therefore, if we set $C(L_K) = 2C_\epsilon(L_K^{\tilde{m}} + L_K^{\tilde{r}} + L_K^{\tilde{q}})$ and $C_0 = C_\epsilon(|f_1(0,0)|^{\tilde{m}}|\Omega| + |f_2(0,0)|^{\tilde{r}}|\Omega| + |h(0)|^{\tilde{q}}|\Gamma|) + \alpha(2|\Omega| + |\Gamma|) + 3\tilde{C}$, then it follows from (2.36) that

$$
E(t) \leq (E(0) + C_0 T_0) + C(L_K) \int_0^t E(\tau) d\tau, \text{ for all } t \in [0, T_0],
$$

where T_0 will be chosen below. By Gronwall's inequality, one has

(2.37) $\qquad E(t) \leq (E(0) + C_0 T_0) e^{C(L_K)t}$ for all $t \in [0, T_0]$.

We select

(2.38) $\qquad T_0 = \min \left\{ \dfrac{1}{4C_0}, \dfrac{1}{C(L_K)} \log 2 \right\},$

and recall our assumption that $K^2 \geq 4E(0) + 1$. Then, it follows from (2.37) that

(2.39) $\qquad E(t) \leq 2(E(0) + 1/4) \leq K^2/2,$

for all $t \in [0, T_0]$. This implies that $\|(u(t), v(t))\|_V \leq K$, for all $t \in [0, T_0]$, and therefore, $f_1^K(u,v) = f_1(u,v)$, $f_2^K(u,v) = f_2(u,v)$ and $h^K(u) = h(\gamma u)$ on the time interval $[0, T_0]$. Because of the uniqueness of solutions for the (K) problem, the solution to the truncated problem (K) coincides with the solution to the system (2.13) for $t \in [0, T_0]$, completing the proof of Lemma 2.2. $\qquad \square$

Remark 2.3. In Lemma 2.2, the local existence time T_0 depends on L_K, which is the local Lipschitz constant of: $f_1 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\frac{m+1}{m}}(\Omega)$, $f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\frac{r+1}{r}}(\Omega)$ and $h(\gamma u) : H^1(\Omega) \longrightarrow L^{\frac{q+1}{q}}(\Gamma)$. The advantage of this result is that T_0 does not depend on the locally Lipschitz constants for the mapping $f_1, f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ and $h(\gamma u) : H^1(\Omega) \longrightarrow L^2(\Gamma)$. This fact is critical for the remaining parts of the proof of the local existence statement in Theorem 1.3.

2.4. **Lipschitz approximations of the sources.** This subsection is devoted to constructing Lipschitz approximations of the sources. The following propositions are needed.

Proposition 2.4. *Assume* $1 \leq p < 6$, $m, r \geq 1$, $p\frac{m+1}{m} \leq \frac{6}{1+2\epsilon}$, *and* $p\frac{r+1}{r} \leq \frac{6}{1+2\epsilon}$, *for some* $\epsilon > 0$. *Further assume that* f_1, $f_2 \in C^1(\mathbb{R}^2)$ *such that*

$$(2.40) \qquad |\nabla f_j(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1),$$

for $j = 1, 2$ *and all* $u, v \in \mathbb{R}$. *Then,* $f_j : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^\sigma(\Omega)$ *is locally Lipschitz continuous,* $j = 1, 2$, *where* $\sigma = \frac{m+1}{m}$ *or* $\sigma = \frac{r+1}{r}$.

Remark 2.5. Since $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$, then it follows from Proposition 2.4 that each f_j is locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^{\frac{m+1}{m}}(\Omega)$ or $L^{\frac{r+1}{r}}(\Omega)$. In particular, if $1 \leq p \leq 3$, then it is easy to verify that each f_j is locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$.

Proof. It is enough to prove that $f_1 : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz continuous, where $\tilde{m} = \frac{m+1}{m}$. Let $(u,v), (\hat{u}, \hat{v}) \in \tilde{V} := H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)$ such that $\|(u,v)\|_{\tilde{V}}, \|(\hat{u}, \hat{v})\|_{\tilde{V}} \leq R$, where $R > 0$. By (2.40) and the mean value theorem, we have

$$|f_1(u,v) - f_1(\hat{u}, \hat{v})|$$
$$(2.41) \qquad \leq C\Big(|u - \hat{u}| + |v - \hat{v}|\Big)\Big(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1\Big).$$

Therefore,

$$\|f_1(u,v) - f_1(\hat{u}, \hat{v})\|_{\tilde{m}}^{\tilde{m}} = \int_\Omega |f_1(u,v) - f_1(\hat{u}, \hat{v})|^{\tilde{m}} dx$$
$$\leq C \int_\Omega \left(|u - \hat{u}|^{\tilde{m}} + |v - \hat{v}|^{\tilde{m}}\right)$$
$$(2.42) \qquad \Big(|u|^{(p-1)\tilde{m}} + |v|^{(p-1)\tilde{m}} + |\hat{u}|^{(p-1)\tilde{m}} + |\hat{v}|^{(p-1)\tilde{m}} + 1\Big) dx.$$

All terms in (2.42) are estimated in the same manner. In particular, for a typical term in (2.42), we estimate it by Hölder's inequality and the Sobolev imbedding $H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$, together with the assumption $p\tilde{m} \leq \frac{6}{1+2\epsilon}$ and $\|u\|_{H^{1-\epsilon}(\Omega)} \leq R$. For instance,

$$\int_\Omega |u - \hat{u}|^{\tilde{m}} |u|^{(p-1)\tilde{m}} dx \leq \left(\int_\Omega |u - \hat{u}|^{p\tilde{m}} dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{p\tilde{m}} dx\right)^{\frac{p-1}{p}}$$
$$\leq C \|u - \hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} \|u\|_{H^{1-\epsilon}(\Omega)}^{(p-1)\tilde{m}} \leq CR^{(p-1)\tilde{m}} \|u - \hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}.$$

Hence, we obtain

$$\|f_1(u,v) - f_1(\hat{u}, \hat{v})\|_{\tilde{m}} \leq C(R) \|(u - \hat{u}, v - \hat{v})\|_{H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)},$$

completing the proof. $\qquad \square$

Recall that for the values $3 < p < 6$, the sources $f_1(u,v)$ and $f_2(u,v)$ are not locally Lipschitz continuous from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$. So, in order to apply Lemma 2.2 to prove Theorem 1.3, we shall construct Lipschitz approximations of the sources f_1 and f_2. In particular, we shall use smooth cutoff functions $\eta_n \in C_0^\infty(\mathbb{R}^2)$,

similar to those used in [30], such that each η_n satisfies: $0 \le \eta_n \le 1$; $\eta_n(u,v) = 1$ if $|(u,v)| \le n$; $\eta_n(u,v) = 0$ if $|(u,v)| \ge 2n$; and $|\nabla \eta_n(u,v)| \le C/n$. Put

$$(2.43) \qquad f_j^n(u,v) = f_j(u,v)\eta_n(u,v), \quad u,v \in \mathbb{R}, \ j=1,2, \ n \in \mathbb{N},$$

where f_1 and f_2 satisfy Assumption 1.1. The following proposition summarizes important properties of f_1^n and f_2^n.

Proposition 2.6. *For each $j=1,2$, $n \in \mathbb{N}$, then function f_j^n, defined in (2.43), satisfies:*

- *$f_j^n(u,v) : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant depending on n.*
- *There exists $\epsilon > 0$ such that $f_j^n : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^\sigma(\Omega)$ is locally Lipschitz continuous where the local Lipschitz constant is independent of n, and where $\sigma = \frac{m+1}{m}$ or $\sigma = \frac{r+1}{r}$.*

Proof. It is enough to prove the proposition for the function f_1^n. Let $(u,v), (\hat{u}, \hat{v}) \in H^1(\Omega) \times H_0^1(\Omega)$ and put

$$\Omega_1 = \{x \in \Omega : |(u(x), v(x))| < 2n, |(\hat{u}(x), \hat{v}(x))| < 2n\},$$
$$\Omega_2 = \{x \in \Omega : |(u(x), v(x))| < 2n, |(\hat{u}(x), \hat{v}(x))| \ge 2n\},$$
$$(2.44) \qquad \Omega_3 = \{x \in \Omega : |(u(x), v(x))| \ge 2n, |(\hat{u}(x), \hat{v}(x))| < 2n\}.$$

By the definition of η, it is clear that $f_1^n(u,v) = f_1^n(\hat{u}, \hat{v}) = 0$ if $|(u,v)| \ge 2n$ and $|(\hat{u}, \hat{v})| \ge 2n$. Therefore, by (2.43) we have

$$(2.45) \qquad \|f_1^n(u,v) - f_1^n(\hat{u}, \hat{v})\|_2^2 = I_1 + I_2 + I_3,$$

where $I_j = \int_{\Omega_j} |f_1(u,v)\eta_n(u,v) - f_1(\hat{u}, \hat{v})\eta_n(\hat{u}, \hat{v})|^2 dx$, $j = 1,2,3$.

Notice that

$$I_1 \le 2\int_{\Omega_1} |f_1(u,v)|^2 |\eta_n(u,v) - \eta_n(\hat{u}, \hat{v})|^2 dx$$
$$(2.46) \qquad + 2\int_{\Omega_1} |\eta_n(\hat{u}, \hat{v})|^2 |f_1(u,v) - f_1(\hat{u}, \hat{v})|^2 dx.$$

Since $|\nabla f_1(u,v)| \le C(|u|^{p-1} + |v|^{p-1} + 1)$, we have

$$(2.47) \qquad |f_1(u,v)| \le C(|u|^p + |v|^p + 1),$$

and along with the fact $|u|, |v| \le 2n$ in Ω_1 and $|\nabla \eta_n| \le C/n$, we obtain

$$\int_{\Omega_1} |f_1(u,v)|^2 |\eta_n(u,v) - \eta_n(\hat{u}, \hat{v})|^2 dx$$
$$\le C\int_{\Omega_1} (|u|^p + |v|^p + 1)^2 |\nabla \eta_n(\xi_1, \xi_2)|^2 |(u - \hat{u}, v - \hat{v})|^2 dx$$
$$(2.48) \qquad \le Cn^{2p-2} \int_{\Omega_1} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx.$$

Moreover, since $|\eta_n| \leq 1$ and $|u|, |\hat{u}|, |v|, |\hat{v}| \leq 2n$ in Ω_1, then by (2.41) we deduce

$$\int_{\Omega_1} |\eta_n(\hat{u}, \hat{v})|^2 |f_1(u, v) - f_1(\hat{u}, \hat{v})|^2 dx$$

$$\leq C \int_{\Omega_1} (|u - \hat{u}|^2 + |v - \hat{v}|^2) \left(|u|^{p-1} + |v|^{p-1} + |\hat{u}|^{p-1} + |\hat{v}|^{p-1} + 1 \right)^2 dx$$

$$(2.49) \quad \leq Cn^{2p-2} \int_{\Omega_1} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx.$$

Therefore, it follows from (2.46), (2.48) and (2.49) that

$$I_1 \leq C(n) \int_{\Omega_1} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx,$$

where $C(n) = Cn^{2p-2}$. To estimate I_2, we note that $\eta_n(\hat{u}, \hat{v}) = 0$ in Ω_2. Then a similar argument as in (2.48) yields

(2.50)
$$I_2 = \int_{\Omega_2} |f_1(u, v)|^2 |\eta_n(u, v) - \eta_n(\hat{u}, \hat{v})|^2 dx \leq C(n) \int_{\Omega_2} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx,$$

where $C(n)$ is as in (2.49). By reversing the roles of (u, v) and $(\hat{u}, \hat{v})$, one also obtains $I_3 \leq C(n) \int_{\Omega_3} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx$. Thus it follows that

$$\|f_1^n(u, v) - f_1^n(\hat{u}, \hat{v})\|_2^2 \leq C(n)(\|u - \hat{u}\|_2^2 + \|v - \hat{v}\|_2^2)$$
$$\leq C(n) \|(u - \hat{u}, v - \hat{v})\|_{H^1(\Omega) \times H_0^1(\Omega)}^2,$$

where $C(n) = Cn^{2p-2}$, which completes the proof of the first statement of the proposition.

To prove the second statement we recall Assumption 1.1, in particular, $p\frac{m+1}{m} < 6$. Then, there exists $\epsilon > 0$ such that $p\frac{m+1}{m} \leq \frac{6}{1+2\epsilon}$. Let $(u, v), (\hat{u}, \hat{v}) \in \tilde{V} := H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)$ such that $\|(u, v)\|_{\tilde{V}}, \|(\hat{u}, \hat{v})\|_{\tilde{V}} \leq R$, where $R > 0$, and recall the notation $\tilde{m} = \frac{m+1}{m}$. Then,

$$(2.51) \qquad \|f_1^n(u, v) - f_1^n(\hat{u}, \hat{v})\|_{\tilde{m}}^{\tilde{m}} = P_1 + P_2 + P_3,$$

where

$$P_j = \int_{\Omega_j} |f_1(u, v)\eta_n(u, v) - f_1(\hat{u}, \hat{v})\eta_n(\hat{u}, \hat{v})|^{\tilde{m}} dx, \quad j = 1, 2, 3,$$

and each Ω_j is as defined in (2.44). Since $|\eta_n| \leq 1$, one has

$$P_1 \leq C \int_{\Omega_1} |f_1(u, v)|^{\tilde{m}} |\eta_n(u, v) - \eta_n(\hat{u}, \hat{v})|^{\tilde{m}} dx$$

$$+ C \int_{\Omega_1} |\eta_n(\hat{u}, \hat{v})|^{\tilde{m}} |f_1(u, v) - f_1(\hat{u}, \hat{v})|^{\tilde{m}} dx$$

$$(2.52) \quad \leq C \int_{\Omega_1} |f_1(u, v)|^{\tilde{m}} |\eta_n(u, v) - \eta_n(\hat{u}, \hat{v})|^{\tilde{m}} dx + C \|f_1(u, v) - f_1(\hat{u}, \hat{v})\|_{\tilde{m}}^{\tilde{m}}.$$

By (2.47) and the mean value theorem, we obtain

$$\int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx$$

$$\le C \int_{\Omega_1} (|u|^p + |v|^p + 1)^{\tilde{m}} |\nabla \eta_n(\xi_1, \xi_2)|^{\tilde{m}} |(u - \hat{u}, v - \hat{v})|^{\tilde{m}} dx$$

$$(2.53) \qquad \le C \int_{\Omega_1} (|u|^{(p-1)\tilde{m}} + |v|^{(p-1)\tilde{m}} + 1)(|u - \hat{u}|^{\tilde{m}} + |v - \hat{v}|^{\tilde{m}}) dx,$$

where we have used the facts $|u|, |v| \le 2n$ in Ω_1 and $|\nabla \eta_n| \le C/n$.

All terms in (2.53) are estimated in the same manner. By using Hölder's inequality, the Sobolev imbedding $H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$, together with the assumption $p\tilde{m} \le \frac{6}{1+2\epsilon}$ and $\|u\|_{H^{1-\epsilon}(\Omega)} \le R$, we obtain

$$\int_{\Omega_1} |u|^{(p-1)\tilde{m}} |u - \hat{u}|^{\tilde{m}} dx \le \left(\int_{\Omega_1} |u|^{p\tilde{m}} dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_1} |u - \hat{u}|^{p\tilde{m}} \right)^{\frac{1}{p}}$$

$$(2.54) \qquad \le C \|u\|_{H^{1-\epsilon}(\Omega)}^{(p-1)\tilde{m}} \|u - \hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} \le C R^{(p-1)\tilde{m}} \|u - \hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}.$$

Therefore, it is easy to see that

$$(2.55) \qquad \int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx \le C(R) \|(u - \hat{u}, v - \hat{v})\|_{\tilde{V}}^{\tilde{m}}.$$

By Proposition 2.4, we know $f_1 : \tilde{V} = H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz. Therefore, it follows from (2.52) and (2.55) that

$$P_1 \le C(R) \|(u - \hat{u}, v - \hat{v})\|_{\tilde{V}}^{\tilde{m}}.$$

To estimate P_2, we use $\eta_n(\hat{u}, \hat{v}) = 0$ in Ω_2 and adopt the same computation in (2.53)-(2.55). Thus, we deduce

$$P_2 = \int_{\Omega_2} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx \le C(R) \|(u - \hat{u}, v - \hat{v})\|_{\tilde{V}}^{\tilde{m}}.$$

Likewise, $P_3 \le C(R) \|(u - \hat{u}, v - \hat{v})\|_{\tilde{V}}^{\tilde{m}}$. Therefore, by (2.51) we have

$$\|f_1^n(u,v) - f_1^n(\hat{u},\hat{v})\|_{\tilde{m}}^{\tilde{m}} \le C(R) \|(u - \hat{u}, v - \hat{v})\|_{\tilde{V}}^{\tilde{m}},$$

where the local Lipschitz constant $C(R)$ is independent of n. This completes the proof of the proposition. $\qquad \square$

The following proposition deals with the boundary source h.

Proposition 2.7. *Assume* $1 \le k < 4$, $q \ge 1$ *and* $k\frac{q+1}{q} \le \frac{4}{1+2\epsilon}$, *for some* $\epsilon > 0$. *If* $h \in C^1(\mathbb{R})$ *such that* $|h'(s)| \le C(|s|^{k-1} + 1)$, *then* $h \circ \gamma$ *is locally Lipschitz:* $H^{1-\epsilon}(\Omega) \longrightarrow L^{\frac{q+1}{q}}(\Gamma)$.

Proof. The proof is very similar to the proof of Proposition 2.4 and it is omitted. $\quad \square$

Remark 2.8. Since $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$, then by Proposition 2.7, we know $h \circ \gamma$ is locally Lipschitz from $H^1(\Omega)$ into $L^{\frac{q+1}{q}}(\Gamma)$. In particular, if $1 \le k \le 2$, we can directly verify that $h \circ \gamma$ is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$.

We note here that if $2 < k < 4$, then $h \circ \gamma$ is not locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Gamma)$. As we have done for the interior sources, we shall construct Lipschitz approximations for the boundary source h. Let $\zeta_n \in C_0^\infty(\mathbb{R})$ be a cutoff function such that $0 \leq \zeta_n \leq 1$; $\zeta_n(s) = 1$ if $|s| \leq n$; $\zeta_n(s) = 0$ if $|s| \geq 2n$; and $|\zeta_n'(s)| \leq C/n$. Put

$$(2.56) \qquad\qquad h^n(s) = h(s)\zeta_n(s), \ s \in \mathbb{R}, \ n \in \mathbb{N},$$

where h satisfies Assumption 1.1. The following proposition summarizes some important properties of h^n.

Proposition 2.9. *For each* $n \in \mathbb{N}$, *the function* h^n *defined in* (2.56) *has the following properties:*

- $h^n \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ *is globally Lipschitz continuous with Lipschitz constant depending on* n.
- *There exists* $\epsilon > 0$ *such that* $h^n \circ \gamma : H^{1-\epsilon}(\Omega) \longrightarrow L^{\frac{q+1}{q}}(\Gamma)$ *is locally Lipschitz continuous where the local Lipschitz constant does not depend on* n.

Proof. The proof is similar to the proof of Proposition 2.6 and it is omitted. $\qquad\square$

2.5. **Approximate solutions and passage to the limit.** We complete the proof of the local existence statement in Theorem 1.3 with the following four steps.

Step 1: *Approximate system.* Recall that in Lemma 2.2, the boundary damping g is assumed *strongly* monotone. However, in Assumption 1.1, we only impose the monotonicity condition on g. To remedy this, we approximate the boundary damping with

$$(2.57) \qquad\qquad g^n(s) = g(s) + \frac{1}{n}s, \ n \in \mathbb{N}.$$

Note that g^n is strongly monotone with the constant $m_g = \frac{1}{n} > 0$, since g is monotone increasing. Indeed, for all $s_1, s_2 \in \mathbb{R}$,

$$(g^n(s_1) - g^n(s_2))(s_1 - s_2) = (g(s_1) - g(s_2))(s_1 - s_2) + \frac{1}{n}|s_1 - s_2|^2 \geq \frac{1}{n}|s_1 - s_2|^2.$$

Corresponding to g^n, we define the operator $\mathcal{S}^n$ as follows: replace g with g^n in (2.7) to define the functional J^n like J in (2.6), and then, similar to (2.9), we define the operator $\mathcal{S}^n : \mathcal{D}(\mathcal{S}^n) = \mathcal{D}(\partial J^n) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ such that $\partial J^n(u) = \{\mathcal{S}^n(u)\}$. As in (2.10) and (2.11), we have for all $u \in \mathcal{D}(\mathcal{S}^n)$,

$$(2.58) \qquad\qquad \langle \mathcal{S}^n(u), u \rangle = \int_\Omega g_1(u)u\,dx + \int_\Gamma g^n(\gamma u)\gamma u\,d\Gamma$$

and

$$(2.59) \qquad \langle \mathcal{S}^n(u), v \rangle = \int_\Omega g_1(u)v\,dx + \int_\Gamma g^n(\gamma u)\gamma v\,d\Gamma \text{ for all } v \in C(\overline{\Omega}).$$

Recall $H = H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, and the approximate sources f_1^n, f_2^n, h^n which were introduced in (2.43) and (2.56). Now, we define the nonlinear operator $\mathscr{A}^n : \mathcal{D}(\mathscr{A}^n) \subset H \longrightarrow H$ by

$$(2.60) \qquad \mathscr{A}^n \begin{bmatrix} u \\ v \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ \Delta_R(u - Rh^n(\gamma u)) + \mathcal{S}^n(y) - f_1^n(u, v) \\ -\Delta v + g_2(z) - f_2^n(u, v) \end{bmatrix}^{tr},$$

where $\mathcal{D}(\mathscr{A}^n) = \Big\{(u, v, y, z) \in \Big(H^1(\Omega) \times H_0^1(\Omega)\Big)^2 : \Delta_R(u - Rh^n(\gamma u)) + \mathcal{S}^n(y) - f_1^n(u, v) \in L^2(\Omega), y \in \mathcal{D}(\mathcal{S}^n), -\Delta v + g_2(z) - f_2^n(u, v) \in L^2(\Omega), g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\Big\}$.

Clearly, from the space of test functions $\mathscr{D}(\Omega)^4 \subset \mathcal{D}(\mathscr{A}^n)$, and since $\mathscr{D}(\Omega)^4$ is dense in H, for each $U_0 = (u_0, v_0, u_1, v_1) \in H$ there exists a sequence of functions $U_0^n = (u_0^n, v_0^n, u_1^n, v_1^n) \in \mathscr{D}(\Omega)^4$ such that $U_0^n \longrightarrow U_0$ in H.

Put $U = (u, v, u_t, v_t)$ and consider the approximate system

$$(2.61) \qquad U_t + \mathscr{A}^n U = 0 \text{ with } U(0) = (u_0^n, v_0^n, u_1^n, v_1^n) \in \mathscr{D}(\Omega)^4.$$

Step 2: Approximate solutions. Since g^n, f_1^n, f_2^n and h^n satisfy the assumptions of Lemma 2.2, then for each n, the approximate problem (2.61) has a strong local solution $U^n = (u^n, v^n, u_t^n, v_t^n) \in W^{1,\infty}(0, T_0; H)$ such that $U^n(t) \in \mathcal{D}(\mathscr{A}^n)$ for $t \in [0, T_0]$. It is important to note here that T_0 is totally independent of n. In fact, by (2.38), T_0 does not depend on the strong monotonicity constant $m_g = \frac{1}{n}$, and although T_0 depends on the local Lipschitz constants of the mappings $f_1^n : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, $f_2^n : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$ and $h^n \circ \gamma : H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$, it is fortunate that these Lipschitz constants are independent of n, thanks to Propositions 2.6 and 2.9. Also, recall that T_0 depends on K, which itself depends on the initial data, and since $U_0^n \to U_0$ in H, we can choose K sufficiently large such that K is uniform for all n. Thus, we will only emphasize the dependence of T_0 on K.

Now, by (2.39), we know $E^n(t) \leq K^2/2$ for all $t \in [0, T_0]$, which implies that

$$(2.62) \qquad \|U^n(t)\|_H^2 = \|u^n(t)\|_{1,\Omega}^2 + \|v^n(t)\|_{1,\Omega}^2 + \|u_t^n(t)\|_2^2 + \|v_t^n(t)\|_2^2 \leq K^2,$$

for all $t \in [0, T_0]$. In addition, by letting $0 < \epsilon \leq \alpha/2$ in (2.35) and by the fact that $\tilde{m}, \tilde{q}, \tilde{r} \leq 2$ and the bound (2.62), we deduce that

$$(2.63) \qquad \int_0^{T_0} \|u_t^n\|_{m+1}^{m+1} dt + \int_0^{T_0} \|v_t^n\|_{r+1}^{r+1} dt + \int_0^{T_0} |\gamma u_t^n|_{q+1}^{q+1} dt < C(K),$$

for some constant $C(K) > 0$. Since $|g_1(s)| \leq \beta |s|^m$ for $|s| \geq 1$ and g_1 is increasing with $g_1(0) = 0$, then $|g_1(s)| \leq \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$. Hence, it follows from (2.63) that

$$(2.64) \qquad \int_0^{T_0} \int_\Omega |g_1(u_t^n)|^{\tilde{m}} dx dt \leq \beta^{\tilde{m}} \int_0^{T_0} \int_\Omega (|u_t^n|^{m+1} + 1) dx dt < C(K).$$

Similarly, one has

$$(2.65) \qquad \int_0^{T_0} \int_\Omega |g_2(v_t^n)|^{\tilde{r}} dx dt < C(K) \text{ and } \int_0^{T_0} \int_\Omega |g^n(\gamma u_t^n)|^{\tilde{q}} dx dt < C(K).$$

Next, we shall prove the following statement: If $w \in H^1(\Omega) \cap L^{m+1}(\Omega)$ with $\gamma w \in L^{q+1}(\Gamma)$, then

$$(2.66) \qquad \langle \mathcal{S}^n(u_t^n), w \rangle = \int_\Omega g_1(u_t^n) w \, dx + \int_\Gamma g^n(\gamma u_t^n) \gamma w \, d\Gamma, \text{ a.e. } [0, T_0].$$

Indeed, by Lemma 3.1 in [4], there exists a sequence $\{w_k\} \subset H^2(\Omega)$ such that $w_k \longrightarrow w$ in $H^1(\Omega)$, $|w_k|^{m+1} \longrightarrow |w|^{m+1}$ in $L^1(\Omega)$ and $|\gamma w_k|^{q+1} \longrightarrow |\gamma w|^{q+1}$ in $L^1(\Gamma)$. By the Generalized Dominated Convergence Theorem, we conclude, on a subsequence labeled the same as $\{w_k\}$,

$$(2.67) \qquad w_k \longrightarrow w \text{ in } L^{m+1}(\Omega) \text{ and } \gamma w_k \longrightarrow \gamma w \text{ in } L^{q+1}(\Gamma).$$

Since $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ (in 3D), and the fact that $u_t^n \in \mathcal{D}(\mathcal{S}^n)$, then it follows from (2.59) that

$$(2.68) \qquad \langle \mathcal{S}^n(u_t^n), w_k \rangle = \int_\Omega g_1(u_t^n) w_k dx + \int_\Gamma g^n(\gamma u_t^n) \gamma w_k d\Gamma.$$

From (2.64) and (2.65) we note that $\|g_1(u_t^n)\|_{\tilde{m}}$ and $|g^n(\gamma u_t^n)|_{\tilde{q}} < \infty$, a.e. $[0, T_0]$. Therefore, by using (2.67), we can pass to the limit in (2.68) as $k \longrightarrow \infty$ to obtain (2.66), as claimed.

Recall that $U^n = (u^n, v^n, u_t^n, v_t^n) \in \mathcal{D}(\mathscr{A}^n)$ is a strong solution of (2.61). If ϕ and ψ satisfy the conditions imposed on test functions in Definition 1.2, then by (2.64)-(2.66), we can test the approximate system (2.61) against ϕ and ψ to obtain

$$(u_t^n(t), \phi(t))_\Omega - (u_1^n, \phi(0))_\Omega - \int_0^t (u_t^n, \phi_t)_\Omega d\tau + \int_0^t (u^n, \phi)_{1,\Omega} d\tau$$

$$+ \int_0^t \int_\Omega g_1(u_t^n) \phi dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t^n) \gamma \phi d\Gamma d\tau + \frac{1}{n} \int_0^t \int_\Gamma \gamma u_t^n \gamma \phi d\Gamma d\tau$$

$$(2.69) \quad = \int_0^t \int_\Omega f_1^n(u^n, v^n) \phi dx d\tau + \int_0^t \int_\Gamma h^n(\gamma u^n) \gamma \phi d\Gamma d\tau,$$

and

$$(v_t^n(t), \psi(t))_\Omega - (v_1^n, \psi(0))_\Omega - \int_0^t (v_t^n, \psi_t)_\Omega d\tau + \int_0^t (v^n, \psi)_{1,\Omega} d\tau$$

$$(2.70) \qquad + \int_0^t \int_\Omega g_2(v_t^n) \psi dx d\tau = \int_0^t \int_\Omega f_2^n(u^n, v^n) \psi dx d\tau$$

for all $t \in [0, T_0]$.

Step 3: *Passage to the limit.* We aim to prove that there exists a subsequence of $\{U^n\}$, labeled again as $\{U^n\}$, that converges to a solution of the original problem (1.1). In what follows, we focus on passing to the limit in (2.69) only, since passing to the limit in (2.70) is similar and is in fact simpler.

First, we note that (2.62) shows $\{U^n\}$ is bounded in $L^\infty(0, T_0; H)$. So, by Alaoglu's Theorem, there exists a subsequence, labeled by $\{U^n\}$, such that

$$(2.71) \qquad U^n \longrightarrow U \text{ weakly}^* \text{ in } L^\infty(0, T_0; H).$$

Also, by (2.62), we know $\{u^n\}$ is bounded in $L^\infty(0, T_0; H^1(\Omega))$, and so $\{u^n\}$ is bounded in $L^s(0, T_0; H^1(\Omega))$ and for any $s > 1$. In addition, by (2.63), we know $\{u_t^n\}$ is bounded in $L^{m+1}(\Omega \times (0, T_0))$, and since $m \geq 1$, we see that $\{u_t^n\}$ is also bounded in $L^{\tilde{m}}(\Omega \times (0, T_0)) = L^{\tilde{m}}(0, T_0; L^{\tilde{m}}(\Omega))$. We note here that for sufficiently small $\epsilon > 0$, the imbedding $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, and $H^{1-\epsilon}(\Omega) \hookrightarrow L^{\tilde{m}}(\Omega)$ (since $\tilde{m} \leq 2$). If $s > 1$ is fixed, then by Aubin's Compactness Theorem, there exists a subsequence such that

$$(2.72) \qquad u^n \longrightarrow u \text{ strongly in } L^s(0, T_0; H^{1-\epsilon}(\Omega)).$$

Similarly, we deduce that there exists a subsequence such that

$$(2.73) \qquad v^n \longrightarrow v \text{ strongly in } L^s(0, T_0; H^{1-\epsilon}(\Omega)).$$

Now, fix $t \in [0, T_0]$. Since $\phi \in C([0, t]; H^1(\Omega))$ and $\phi_t \in L^1(0, t; L^2(\Omega))$, then by (2.71), we obtain

$$(2.74) \qquad \lim_{n \longrightarrow \infty} \int_0^t (u^n, \phi)_{1,\Omega} dx d\tau = \int_0^t (u, \phi)_{1,\Omega} dx d\tau$$

and

$$(2.75) \qquad \lim_{n \longrightarrow \infty} \int_0^t (u_t^n, \phi_t)_\Omega dx d\tau = \int_0^t (u_t, \phi_t)_\Omega dx d\tau.$$

In addition, since $\tilde{q} \leq 2 \leq q + 1$ and $\gamma\phi \in L^{q+1}(\Gamma \times (0, t))$, then $\gamma\phi \in L^{\tilde{q}}(\Gamma \times (0, t))$, and along with (2.63), one has

$$(2.76) \quad \left| \frac{1}{n} \int_0^t \int_\Gamma \gamma u_t^n \gamma\phi d\Gamma d\tau \right| \leq \frac{1}{n} \left(\int_0^t |\gamma u_t^n|_{q+1}^{q+1} d\tau \right)^{\frac{1}{q+1}} \left(\int_0^t |\gamma\phi|_{\tilde{q}}^{\tilde{q}} dt \right)^{\frac{q}{q+1}} \longrightarrow 0.$$

Moreover, by (2.64)-(2.65), on a subsequence,

$$(2.77) \qquad \begin{cases} g_1(u_t^n) \longrightarrow g_1^* \text{ weakly in } L^{\tilde{m}}(\Omega \times (0, t)), \\ g(\gamma u_t^n) \longrightarrow g^* \text{ weakly in } L^{\tilde{q}}(\Gamma \times (0, t)), \end{cases}$$

for some $g_1^* \in L^{\tilde{m}}(\Omega \times (0, t))$ and some $g^* \in L^{\tilde{q}}(\Gamma \times (0, t))$. Our goal is to show that $g_1^* = g_1(u_t)$ and $g^* = g(\gamma u_t)$. In order to do so, we consider two solutions to the approximate problem (2.61), U^n and U^j. For the sake of simplifying the notation, put $\tilde{u} = u^n - u^j$. Since $U^n, U^j \in W^{1,\infty}(0, T_0; H)$ and $U^n(t), U^j(t) \in \mathcal{D}(\mathscr{A}^n)$, then $\tilde{u}_t \in W^{1,\infty}(0, T_0; L^2(\Omega))$ and $\tilde{u}_t(t) \in H^1(\Omega)$. Moreover, by (2.63) we know $\tilde{u}_t \in L^{m+1}(\Omega \times (0, T_0))$ and $\gamma\tilde{u}_t \in L^{q+1}(\Gamma \times (0, T_0))$. Hence, we may consider the difference of the approximate problems corresponding to the parameters n and j, and then use the multiplier $\tilde{u}_t$ on the first equation. By performing integration by parts in the first equation, one has the following energy identity:

$$\frac{1}{2} \left(\|\tilde{u}_t(t)\|_2^2 + \|\tilde{u}(t)\|_{1,\Omega}^2 \right) + \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))\tilde{u}_t dx d\tau$$

$$+ \int_0^t \int_\Gamma (g(\gamma u_t^n) - g(\gamma u_t^j))\gamma\tilde{u}_t d\Gamma d\tau + \int_0^t \int_\Gamma \left(\frac{1}{n}\gamma u_t^n - \frac{1}{j}\gamma u_t^j \right) \gamma\tilde{u}_t d\Gamma d\tau$$

$$= \frac{1}{2} \left(\|\tilde{u}_t(0)\|_2^2 + \|\tilde{u}(0)\|_{1,\Omega}^2 \right) + \int_0^t \int_\Omega (f_1^n(u^n, v^n) - f_1^j(u^j, v^j))\tilde{u}_t dx d\tau$$

$$(2.78) \qquad + \int_0^t \int_\Gamma (h^n(\gamma u^n) - h^j(\gamma u^j))\gamma\tilde{u}_t d\Gamma d\tau,$$

where we have used (2.66). It follows from (2.78) that

$$
\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2 + \|\tilde{u}(t)\|_{1,\Omega}^2\right) + \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))\tilde{u}_t dx d\tau
$$

$$
+ \int_0^t \int_\Gamma (g(\gamma u_t^n) - g(\gamma u_t^j))\gamma\tilde{u}_t d\Gamma d\tau
$$

$$
\leq \frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2 + \|\tilde{u}(0)\|_{1,\Omega}^2\right) + 2\left(\frac{1}{n}+\frac{1}{j}\right)\int_0^t \int_\Gamma \left(|\gamma u_t^n|^2 + |\gamma u_t^j|^2\right) d\Gamma d\tau
$$

$$
+ \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau
$$

$$
(2.79) \qquad + \int_0^t \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma\tilde{u}_t| d\Gamma d\tau.
$$

We will show that each term on the right hand side of (2.79) converges to 0 as $n, j \longrightarrow \infty$. First, since $\lim_{n\to 0}\|u_0^n - u_0\|_{1,\Omega} = 0$ and $\lim_{n\to 0}\|u_1^n - u_1\|_2 = 0$, we obtain

$$
\lim_{n,j\longrightarrow 0}\|\tilde{u}(0)\|_{1,\Omega} = \lim_{n,j\longrightarrow 0}\left\|u_0^n - u_0^j\right\|_{1,\Omega} = 0,
$$

$$
(2.80) \qquad \lim_{n,j\longrightarrow 0}\|\tilde{u}_t(0)\|_2 = \lim_{n,j\longrightarrow 0}\left\|u_1^n - u_1^j\right\|_2 = 0.
$$

By (2.63), we know $\int_0^t |\gamma u_t^n|_{q+1}^{q+1} d\tau < C(K)$ for all $n \in \mathbb{N}$. Since $q \geq 1$, it is easy to see $\int_0^t |\gamma u_t^n|_2^2 d\tau$ is also uniformly bounded in n. Thus,

$$
(2.81) \qquad \lim_{n,j\longrightarrow\infty}\left(\frac{1}{n}+\frac{1}{j}\right)\int_0^t \int_\Gamma \left(|\gamma u_t^n|^2 + |\gamma u_t^j|^2\right) d\Gamma d\tau = 0.
$$

Next we look at the third term on the right hand side of (2.79). We have

$$
(2.82)
$$
$$
\int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau
$$

$$
\leq \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u, v)||\tilde{u}_t| dx d\tau + \int_0^t \int_\Omega |f_1^n(u, v) - f_1(u, v)||\tilde{u}_t| dx d\tau
$$

$$
+ \int_0^t \int_\Omega |f_1(u, v) - f_1^j(u, v)||\tilde{u}_t| dx d\tau
$$

$$
+ \int_0^t \int_\Omega |f_1^j(u, v) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau.
$$

We now estimate each term on the right hand side of (2.82) as follows. Recall, by Proposition 2.6, $f_1^n : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz, where the local Lipschitz constant is independent of n. By using Hölder's inequality, we

obtain

$$\int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u, v)||\tilde{u}_t| dx d\tau$$

$$\leq \left(\int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u, v)|^{\tilde{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |u_t|^{m+1} dx d\tau \right)^{\frac{1}{m+1}}$$

$$(2.83) \quad \leq C(K) \left(\int_0^t (\|u^n - u\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} + \|v^n - v\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}) d\tau \right)^{\frac{m}{m+1}} \longrightarrow 0,$$

as $n \longrightarrow \infty$, where we have used the convergence (2.72)-(2.73) and the uniform bound in (2.63).

To handle the second term on the right hand side of (2.82), we shall show

$$(2.84) \qquad f_1^n(u, v) \longrightarrow f_1(u, v) \text{ in } L^{\tilde{m}}(\Omega \times (0, T_0)).$$

Indeed, by (2.71), we know $U \in L^\infty(0, T_0; H)$; thus $u \in L^\infty(0, T_0; H^1(\Omega))$ and $v \in L^\infty(0, T_0; H_0^1(\Omega))$. In addition, by (2.43), the definition of f_1^n, we have

$$(2.85) \quad \|f_1^n(u, v) - f_1(u, v)\|_{L^{\tilde{m}}(\Omega \times (0,T_0))}^{\tilde{m}} = \int_0^{T_0} \int_\Omega (|f_1(u, v)||\eta_n(u, v) - 1|)^{\tilde{m}} dx dt.$$

Since $\eta_n(u, v) \leq 1$, it follows that $(|f_1(u, v)||\eta_n(u, v) - 1|)^{\tilde{m}} \leq 2^{\tilde{m}} |f_1(u, v)|^{\tilde{m}}$. To see $|f_1(u, v)|^{\tilde{m}} \in L^1(\Omega \times (0, T_0))$, we use the assumptions $|f_1(u, v)| \leq C(|u|^p + |v|^p + 1)$ and $p\tilde{m} < 6$ along with the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Indeed,

$$\int_0^{T_0} \int_\Omega |f_1(u, v)|^{\tilde{m}} dx dt \leq C \int_0^{T_0} \int_\Omega (|u|^{p\tilde{m}} + |v|^{p\tilde{m}} + 1) dx dt$$

$$\leq C \int_0^{T_0} (\|u\|_{H^1(\Omega)}^{p\tilde{m}} + \|v\|_{H_0^1(\Omega)}^{p\tilde{m}} + |\Omega|) dt < \infty.$$

Clearly, $\eta_n(u(x), v(x)) \longrightarrow 1$ a.e. on Ω. By applying the Lebesgue Dominated Convergence Theorem on (2.85), (2.84) follows, as desired. Now, by using Hölder's inequality and the limit (2.84), one has

$$(2.86)$$

$$\int_0^t \int_\Omega |f_1^n(u, v) - f_1(u, v)||\tilde{u}_t| dx d\tau$$

$$\leq \left(\int_0^t \int_\Omega |f_1^n(u, v) - f_1(u, v)|^{\tilde{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\tilde{u}_t|^{m+1} dx d\tau \right)^{\frac{1}{m+1}} \longrightarrow 0,$$

as $n \longrightarrow \infty$, where we have used the uniform bound in (2.63).

Combining (2.83) and (2.86) in (2.82) gives us the desired result,

$$(2.87) \qquad \lim_{n, j \longrightarrow \infty} \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau = 0.$$

Next we show that

$$(2.88) \qquad \lim_{n, j \longrightarrow \infty} \int_0^t \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma \tilde{u}_t| d\Gamma d\tau = 0.$$

To see this, we write

$$\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma \tilde{u}_t| d\Gamma d\tau$$

$$\leq \int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)||\gamma \tilde{u}_t| d\Gamma d\tau + \int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)||\gamma \tilde{u}_t| d\Gamma d\tau$$

$$(2.89) \quad + \int_0^t \int_\Gamma |h(\gamma u) - h^j(\gamma u)||\gamma \tilde{u}_t| d\Gamma d\tau + \int_0^t \int_\Gamma |h^j(\gamma u) - h^j(\gamma u^j)||\gamma \tilde{u}_t| d\Gamma d\tau.$$

By Proposition 2.9, $h^n \circ \gamma : H^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ is locally Lipschitz, where the local Lipschitz constant is independent of n. Therefore, by Hölder's inequality

$$\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)||\gamma \tilde{u}_t| d\Gamma d\tau$$

$$\leq \left(\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)|^{\tilde{q}} d\Gamma d\tau \right)^{\frac{q}{q+1}} \left(\int_0^t \int_\Gamma |\gamma \tilde{u}_t|^{q+1} d\Gamma d\tau \right)^{\frac{1}{q+1}}$$

$$(2.90) \quad \leq C(K) \left(\int_0^t \|u^n - u\|^{\tilde{q}}_{H^{1-\epsilon}(\Omega)} d\tau \right)^{\frac{q}{q+1}} \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

where we have used the convergence (2.72) and the uniform bound in (2.63).

Since $u \in L^\infty(0, T; H^1(\Omega))$, then similar to (2.84), we may deduce that

$$h^n(\gamma u) \longrightarrow h(\gamma u) \text{ in } L^{\tilde{q}}(\Omega \times (0, T_0)).$$

Again, by using the uniform bound in (2.63), we obtain

$$\int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)||\gamma \tilde{u}_t| d\Gamma d\tau$$

$$(2.91) \quad \leq \left(\int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)|^{\tilde{q}} d\Gamma d\tau \right)^{\frac{q}{q+1}} \left(\int_0^t \int_\Gamma |\gamma \tilde{u}_t|^{q+1} d\Gamma d\tau \right)^{\frac{1}{q+1}} \longrightarrow 0,$$

as $n \longrightarrow \infty$. By combining the estimates (2.89)-(2.91), then (2.88) follows, as claimed.

Now, by using the fact that g_1 and g are monotone increasing and using (2.80)-(2.81), (2.87)-(2.88), we can take limit as $n, j \longrightarrow \infty$ in (2.79) to deduce that

$$(2.92) \qquad \lim_{n,j \longrightarrow \infty} \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))(u_t^n - u_t^j) dx d\tau = 0,$$

$$(2.93) \qquad \lim_{n,j \longrightarrow \infty} \int_0^t \int_\Gamma (g(\gamma u_t^n) - g(\gamma u_t^j))(\gamma u_t^n - \gamma u_t^j) d\Gamma d\tau = 0.$$

In addition, it follows from (2.63) that, on a relabeled subsequence, $u_t^n \longrightarrow u_t$ weakly in $L^{m+1}(\Omega \times (0, T_0))$. Therefore, Lemma 1.3 (p. 49) of [5] along with (2.77) and (2.92) assert that $g_1^* = g_1(u_t)$, provided we show that

$$g_1 : L^{m+1}(\Omega \times (0, t)) \longrightarrow L^{\tilde{m}}(\Omega \times (0, t))$$

is maximal monotone. Indeed, since g_1 is monotone increasing, it is easy to see g_1 is a monotone operator. Thus, we need to verify that g_1 is hemi-continuous, i.e.,

we have to show that

$$(2.94) \qquad \lim_{\lambda \longrightarrow \infty} \int_0^t \int_\Omega g_1(u + \lambda v) w \, dx d\tau = \int_0^t \int_\Omega g_1(u) w \, dx d\tau,$$

for all $u, v, w \in L^{m+1}(\Omega \times (0, t))$.

Indeed, since g_1 is continuous, then $g_1(u + \lambda v)w \longrightarrow g_1(u)w$ point-wise as $\lambda \longrightarrow 0$. Moreover, since $|g_1(s)| \le \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$, we know if $|\lambda| \le 1$, then $|g_1(u + \lambda v)w| \le \beta(|u + \lambda v|^m + 1)|w| \le C(|u|^m|w| + |v|^m|w| + |w|) \in L^1(\Omega \times (0, t))$, by Hölder's inequality. Thus, (2.94) follows from the Lebesgue Dominated Convergence Theorem. Hence, g_1 is maximal monotone and we conclude that $g_1^* = g_1(u_t)$, i.e.,

$$(2.95) \qquad g_1(u_t^n) \longrightarrow g_1(u_t) \text{ weakly in } L^{\tilde{m}}(\Omega \times (0, t)).$$

In a similar way, one can show that $g^* = g(\gamma u_t)$, that is,

$$(2.96) \qquad g(\gamma u_t^n) \longrightarrow g(\gamma u_t) \text{ weakly in } L^{\tilde{q}}(\Gamma \times (0, t)).$$

It remains to show that

$$(2.97) \qquad \lim_{n \longrightarrow \infty} \int_0^t \int_\Omega f_1^n(u^n, v^n) \phi \, dx d\tau = \int_0^t \int_\Omega f_1(u, v) \phi \, dx d\tau.$$

To prove (2.97), we write

(2.98)
$$\left| \int_0^t \int_\Omega (f_1^n(u^n, v^n) - f_1(u, v)) \phi \, dx d\tau \right|$$
$$\le \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u, v)||\phi| dx d\tau + \int_0^t \int_\Omega |f_1^n(u, v) - f_1(u, v)||\phi| dx d\tau.$$

Since $\phi \in L^{m+1}(\Omega \times (0, t))$, then by replacing $\tilde{u}_t$ with ϕ in (2.83), we deduce

$$(2.99) \qquad \lim_{n \longrightarrow \infty} \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u, v)||\phi| dx d\tau = 0.$$

In addition, (2.84) yields

$$(2.100) \qquad \lim_{n \to \infty} \int_0^t \int_\Omega |f_1^n(u, v) - f_1(u, v)||\phi| dx d\tau = 0.$$

Hence, (2.97) is verified.

In a similar manner, one can deduce

$$(2.101) \qquad \lim_{n \longrightarrow \infty} \int_0^t \int_\Gamma h^n(\gamma u^n) \gamma \phi d\Gamma d\tau = \int_0^t \int_\Gamma h(\gamma u) \gamma \phi d\Gamma d\tau.$$

Finally, by using (2.71)-(2.76), (2.95)-(2.97) and (2.101) we can pass to the limit in (2.69) to obtain (1.6). In a similar way, we can work on (2.70) term by term to pass to the limit and obtain (1.7) .

Step 4: *Completion of the proof.* Since $t \in [0, T_0]$ and g, g_1 are monotone increasing on $\mathbb{R}$, then (2.79) implies

$$\frac{1}{2} \left(\|\tilde{u}_t(t)\|_2^2 + \|\tilde{u}(t)\|_{1,\Omega}^2 \right)$$

$$\leq \frac{1}{2} \left(\|\tilde{u}_t(0)\|_2^2 + \|\tilde{u}(0)\|_{1,\Omega}^2 \right) + 2 \left(\frac{1}{n} + \frac{1}{m} \right) \int_0^{T_0} \int_\Gamma (|\gamma u_t^n|^2 + |\gamma u_t^j|^2) d\Gamma d\tau$$

$$+ \int_0^{T_0} \int_\Omega |f_1^n(u^n, v^n) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau$$

$$(2.102) \qquad + \int_0^{T_0} \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma \tilde{u}_t| d\Gamma d\tau.$$

By (2.80)-(2.81) and (2.87)-(2.88), we know the right hand side of (2.102) converges to 0 as $n, j \longrightarrow \infty$, so

$$\lim_{n,j \longrightarrow \infty} \left\| u^n(t) - u^j(t) \right\|_{1,\Omega} = \lim_{n,j \longrightarrow \infty} \|\tilde{u}(t)\|_{1,\Omega} = 0 \text{ uniformly in } t \in [0, T_0],$$

$$\lim_{n,j \longrightarrow \infty} \left\| u_t^n(t) - u_t^j(t) \right\|_2 = \lim_{n,j \longrightarrow \infty} \|\tilde{u}_t(t)\|_2 = 0 \text{ uniformly in } t \in [0, T_0].$$

Hence

$$u^n(t) \longrightarrow u(t) \text{ in } H^1(\Omega) \text{ uniformly on } [0, T_0],$$

$$(2.103) \qquad u_t^n(t) \longrightarrow u_t(t) \text{ in } L^2(\Omega) \text{ uniformly on } [0, T_0].$$

Since $u^n \in W^{1,\infty}([0, T_0]; H^1(\Omega))$ and $u_t^n \in W^{1,\infty}([0, T_0]; L^2(\Omega))$, by (2.103), we conclude

$$u \in C([0, T_0]; H^1(\Omega)) \text{ and } u_t \in C([0, T_0]; L^2(\Omega)).$$

Moreover, (2.103) shows $u^n(0) \longrightarrow u(0)$ in $H^1(\Omega)$. Since $u^n(0) = u_0^n \longrightarrow u_0$ in $H^1(\Omega)$, then the initial condition $u(0) = u_0$ holds. Also, since $u_t^n(0) \longrightarrow u_t(0)$ in $L^2(\Omega)$ and $u_t^n(0) = u_1^n \longrightarrow u_1$ in $L^2(\Omega)$, we obtain $u_t(0) = u_1$. Similarly, we may deduce that v, v_t satisfy the required regularity and the imposed initial conditions, as stated in Definition 1.2. This completes the proof of the local existence statement in Theorem 1.3.

3. ENERGY IDENTITY

This section is devoted to deriving the energy identity (1.8) in Theorem 1.3. One is tempted to test (1.6) with u_t and (1.7) with v_t, and with that carry out standard calculations to obtain energy identity. However, this procedure is only *formal*, since u_t and v_t are not regular enough and cannot be used as test functions in (1.6) and (1.7). In order to overcome this difficulty we shall use the difference quotients $D_h u$ and $D_h v$ and their well-known properties (see [19] and also [33, 36] for more details).

3.1. **Properties of the difference quotient.** Let X be a Banach space. For any function $u \in C([0, T]; X)$ and $h > 0$, we define the *symmetric difference quotient* by

$$(3.1) \qquad D_h u(t) = \frac{u_e(t+h) - u_e(t-h)}{2h},$$

where $u_e(t)$ denotes the extension of $u(t)$ to $\mathbb{R}$ given by

$$
(3.2) \qquad u_e(t) = \begin{cases} u(0) \text{ for } t \leq 0, \\ u(t) \text{ for } t \in (0,T), \\ u(T) \text{ for } t \geq T. \end{cases}
$$

The results in the following proposition have been established by Koch and Lasiecka in [19].

Proposition 3.1 ([19]). *Let* $u \in C([0,T]; X)$, *where* X *is a Hilbert space with inner product* $(\cdot, \cdot)_X$. *Then,*

$$
(3.3) \qquad \lim_{h \to 0} \int_0^T (u, D_h u)_X \, dt = \frac{1}{2} \left(\|u(T)\|_X^2 - \|u(0)\|_X^2 \right).
$$

If, in addition, $u_t \in C([0,T]; X)$, *then*

$$
(3.4) \qquad \int_0^T (u_t, (D_h u)_t)_X \, dt = 0, \text{ for each } h > 0,
$$

and, as $h \longrightarrow 0$,

$$
(3.5) \qquad D_h u(t) \longrightarrow u_t(t) \text{ weakly in } X, \text{ for every } t \in (0,T),
$$

$$
(3.6) \qquad D_h u(0) \longrightarrow \frac{1}{2} u_t(0) \text{ and } D_h u(T) \longrightarrow \frac{1}{2} u_t(T) \text{ weakly in } X.
$$

The following proposition is essential for the proof of the energy identity (1.8).

Proposition 3.2. *Let* X *and* Y *be Banach spaces. Assume* $u \in C([0,T]; Y)$ *and* $u_t \in L^1(0,T; Y) \cap L^p(0,T; X)$, *where* $1 \leq p < \infty$. *Then* $D_h u \in L^p(0,T; X)$ *and* $\|D_h u\|_{L^p(0,T;X)} \leq \|u_t\|_{L^p(0,T;X)}$. *Moreover,* $D_h u \longrightarrow u_t$ *in* $L^p(0,T; X)$, *as* $h \longrightarrow 0$.

Proof. Throughout the proof, we write u_t as u'. Since $u \in C([0,T]; Y)$, then by (3.2), $u_e \in C([-h, T+h]; Y)$. Also note that

$$
(3.7) \qquad u_e'(t) = u'(t) \text{ for } t \in (0,T) \text{ and } u_e'(t) = 0 \text{ for } t \in (-h, 0) \cup (T, T+h),
$$

and along with the assumption $u' \in L^1(0,T; Y)$, one has $u_e' \in L^1(-h, T+h; Y)$. Since u_e and $u_e' \in L^1(-h, T+h; Y)$, we conclude (for instance, see Lemma 1.1, page 250 in [41])

$$
(3.8) \qquad D_h u(t) = \frac{u_e(t+h) - u_e(t-h)}{2h} = \frac{1}{2h} \int_{t-h}^{t+h} u_e'(s) ds, \text{ a.e. } t \in [0,T].
$$

By using Jensen's inequality, it follows that

$$
(3.9) \qquad \|D_h u(t)\|_X^p \leq \frac{1}{2h} \int_{t-h}^{t+h} \|u_e'(s)\|_X^p \, ds, \text{ a.e. } t \in [0,T].
$$

By integrating both sides of (3.9) over $[0,T]$ and by using Tonelli's Theorem, one has

$$
(3.10)
$$
$$
\int_0^T \|D_h u(t)\|_X^p \, dt \leq \frac{1}{2h} \int_0^T \int_{t-h}^{t+h} \|u_e'(s)\|_X^p \, ds dt = \frac{1}{2h} \int_0^T \int_{-h}^{h} \|u_e'(s+t)\|_X^p \, ds dt
$$
$$
= \frac{1}{2h} \int_{-h}^{h} \int_0^T \|u_e'(s+t)\|_X^p \, dt ds = \frac{1}{2h} \int_{-h}^{h} \int_s^{T+s} \|u_e'(t)\|_X^p \, dt ds.
$$

We split the last integral in (3.10) as the sum of two integrals, and by recalling (3.7), we deduce that

$$
\begin{aligned}
\int_0^T \|D_h u(t)\|_X^p \, dt &\leq \frac{1}{2h} \int_{-h}^0 \int_s^{T+s} \|u_e'(t)\|_X^p \, dtds + \frac{1}{2h} \int_0^h \int_s^{T+s} \|u_e'(t)\|_X^p \, dtds \\
&= \frac{1}{2h} \int_{-h}^0 \int_0^{T+s} \|u'(t)\|_X^p \, dtds + \frac{1}{2h} \int_0^h \int_s^T \|u'(t)\|_X^p \, dsdt \\
&\leq \frac{1}{2h} \int_{-h}^0 \int_0^T \|u'(t)\|_X^p \, dtds + \frac{1}{2h} \int_0^h \int_0^T \|u'(t)\|_X^p \, dtds \\
&= \frac{1}{2h} \int_{-h}^h \int_0^T \|u'(t)\|_X^p \, dtds = \int_0^T \|u'(t)\|_X^p \, dt.
\end{aligned}
$$

Thus,

$$
(3.11) \qquad \|D_h u\|_{L^p(0,T;X)} \leq \|u'\|_{L^p(0,T;X)},
$$

as desired.

It remains to show that $D_h u \longrightarrow u'$ in $L^p(0,T;X)$, as $h \longrightarrow 0$.

Let $\epsilon > 0$ be given. By Lemma 7.2 in the Appendix, $C_0((0,T);X)$ is dense in $L^p(0,T;X)$, and since $u' \in L^p(0,T;X)$, there exists $\phi \in C_0((0,T);X)$ such that $\|u' - \phi\|_{L^p(0,T;X)} \leq \epsilon/3$. Note that (3.8) yields

$$
D_h u(t) - u'(t) = \frac{1}{2h} \int_{t-h}^{t+h} (u_e'(s) - u'(t))ds, \text{ a.e. } t \in [0,T].
$$

In particular,

$$
\begin{aligned}
\|D_h u(t) - u'(t)\|_X^p &\leq \frac{1}{2h} \int_{t-h}^{t+h} \|u_e'(s) - u'(t)\|_X^p \, ds \\
&\leq \frac{1}{2h} \int_{t-h}^{t+h} \Big(\|u_e'(s) - \phi(s)\|_X + \|\phi(s) - \phi(t)\|_X + \|\phi(t) - u'(t)\|_X \Big)^p ds \\
&\leq \frac{3^{p-1}}{2h} \int_{t-h}^{t+h} \|u_e'(s) - \phi(s)\|_X^p \, ds + \frac{3^{p-1}}{2h} \int_{t-h}^{t+h} \|\phi(s) - \phi(t)\|_X^p \, ds
\end{aligned}
$$

$$
(3.12) \qquad + 3^{p-1} \|\phi(t) - u'(t)\|_X^p,
$$

where we have used Jensen's inequality. Now, integrating both sides of (3.12) over $[0,T]$ we obtain

$$
(3.13) \qquad \int_0^T \|D_h u(t) - u'(t)\|_X^p \, dt \leq I_1 + I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &= \frac{3^{p-1}}{2h} \int_0^T \int_{t-h}^{t+h} \|u_e'(s) - \phi(s)\|_X^p \, dsdt, \\
I_2 &= \frac{3^{p-1}}{2h} \int_0^T \int_{t-h}^{t+h} \|\phi(s) - \phi(t)\|_X^p \, dsdt, \\
I_3 &= 3^{p-1} \|\phi(t) - u'(t)\|_{L^p(0,T;X)}^p.
\end{aligned}
$$

Since $\|u' - \phi\|_{L^p(0,T;X)} \leq \epsilon/3$, then

$$
(3.14) \qquad I_3 \leq 3^{p-1} \frac{\epsilon^p}{3^p} = \frac{\epsilon^p}{3}.
$$

In addition, since $\phi \in C_0((0,T);X)$, then $\phi : \mathbb{R} \longrightarrow X$ is uniformly continuous. Thus, there exists $\delta > 0$ (say $\delta < T$) such that $\|\phi(s) - \phi(t)\|_X < \frac{\epsilon}{3T^{1/p}}$ whenever $|s - t| < \delta$. So, if $0 < h < \frac{\delta}{2}$, then one has

$$(3.15) \qquad I_2 \leq \frac{3^{p-1}}{2h} \int_0^T \int_{t-h}^{t+h} \left(\frac{\epsilon}{3T^{1/p}} \right)^p ds\, dt = \frac{\epsilon^p}{3}.$$

As for I_1, we change variables and use Tonelli's Theorem as follows:

$$I_1 = \frac{3^{p-1}}{2h} \int_0^T \int_{-h}^h \|u_e'(s+t) - \phi(s+t)\|_X^p \, ds\, dt$$

$$(3.16) \qquad = \frac{3^{p-1}}{2h} \int_{-h}^h \int_s^{T+s} \|u_e'(t) - \phi(t)\|_X^p \, dt\, ds.$$

Now, split I_1 into two integrals and recall (3.7) to obtain (for sufficiently small h)

$$I_1 = \frac{3^{p-1}}{2h} \left(\int_{-h}^0 \int_0^{T+s} \|u'(t) - \phi(t)\|_X^p \, dt\, ds + \int_0^h \int_s^T \|u'(t) - \phi(t)\|_X^p \, dt\, ds \right)$$

$$\leq \frac{3^{p-1}}{2h} \int_{-h}^h \int_0^T \|u'(t) - \phi(t)\|_X^p \, dt\, ds = 3^{p-1} \int_0^T \|u'(t) - \phi(t)\|_X^p \, dt$$

$$(3.17) \quad = 3^{p-1} \|u' - \phi\|_{L^p(0,T;X)}^p \leq 3^{p-1} \cdot \frac{\epsilon^p}{3^p} = \frac{\epsilon^p}{3}.$$

Therefore, if $0 < h < \frac{\delta}{2}$, then it follows from (3.14), (3.15), (3.17), and (3.13) that

$$\|D_h u - u'\|_{L^p(0,T;X)}^p \leq \epsilon^p,$$

completing the proof. $\qquad \square$

3.2. **Proof of the energy identity.** Throughout the proof, we fix $t \in [0, T_0]$ and let (u, v) be a weak solution of system (1.1) in the sense of Definition 1.2. Recall the regularity of u and v, in particular, $u_t \in C([0,t]; L^2(\Omega))$ and $u_t \in L^{m+1}(\Omega \times (0,t)) = L^{m+1}(0,t; L^{m+1}(\Omega))$. We can define the difference quotient $D_h u(\tau)$ on $[0,t]$ as in (3.1), i.e., $D_h u(\tau) = \frac{1}{2h}[u_e(\tau+h) - u_e(\tau-h)]$, where $u_e(\tau)$ extends $u(\tau)$ from $[0,t]$ to $\mathbb{R}$ as in (3.2). By Proposition 3.2, with $X = L^{m+1}(\Omega)$ and $Y = L^2(\Omega)$, we have

$$(3.18) \qquad D_h u \in L^{m+1}(\Omega \times (0,t)) \ \text{ and } \ D_h u \longrightarrow u_t \ \text{ in } \ L^{m+1}(\Omega \times (0,t)).$$

A similar argument yields

$$(3.19) \qquad D_h v \in L^{r+1}(\Omega \times (0,t)) \ \text{ and } \ D_h v \longrightarrow v_t \ \text{ in } \ L^{r+1}(\Omega \times (0,t)).$$

Recall that the notation γu_t stands for $(\gamma u)_t$, and since $u \in C([0,t]; H^1(\Omega))$, then $\gamma u \in C([0,t]; L^2(\Gamma))$. Moreover, we know $(\gamma u)_t = \gamma u_t \in L^{q+1}(\Gamma \times (0,t)) = L^{q+1}(0,t; L^{q+1}(\Gamma))$, so $(\gamma u)_t \in L^2(\Gamma \times (0,t)) = L^2(0,t; L^2(\Gamma))$. So, by Proposition 3.2 with $X = L^{q+1}(\Gamma)$ and $Y = L^2(\Gamma)$, one has

$$\gamma D_h u = D_h(\gamma u) \in L^{q+1}(\Gamma \times (0,t)) \ \text{ and }$$

$$(3.20) \qquad \gamma D_h u = D_h(\gamma u) \longrightarrow (\gamma u)_t = \gamma u_t \ \text{ in } \ L^{q+1}(\Gamma \times (0,t)).$$

Moreover, since $u \in C([0,t]; H^1(\Omega))$ and $v \in C([0,t]; H_0^1(\Omega))$, then

$$(3.21) \qquad D_h u \in C([0,t]; H^1(\Omega)) \ \text{ and } \ D_h v \in C([0,t]; H_0^1(\Omega)).$$

We now show

$$(3.22) \qquad (D_h u)_t \in L^1(0,t; L^2(\Omega)) \ \text{ and } \ (D_h v)_t \in L^1(0,t; L^2(\Omega)).$$

Indeed, for $0 < h < \frac{t}{2}$, we note that

$$(D_h u)_t(\tau) = \begin{cases} \frac{1}{2h}[u_t(\tau + h) - u_t(\tau - h)], & \text{if } h < \tau < t - h, \\ -\frac{1}{2h} u_t(\tau - h), & \text{if } t - h < \tau < t, \\ \frac{1}{2h} u_t(\tau + h), & \text{if } 0 < \tau < h, \end{cases}$$

and since $u_t \in C([0, t]; L^2(\Omega))$, we conclude $(D_h u)_t \in L^1(0, t; L^2(\Omega))$. Similarly, $(D_h v)_t \in L^1(0, t; L^2(\Omega))$.

Thus, (3.18)-(3.22) show that $D_h u$ and $D_h v$ satisfy the required regularity conditions to be suitable test functions in Definition 1.2. Therefore, by taking $\phi = D_h u$ in (1.6) and $\psi = D_h v$ in (1.7), we obtain

$$(u_t(t), D_h u(t))_\Omega - (u_t(0), D_h u(0))_\Omega - \int_0^t (u_t, (D_h u)_t)_\Omega d\tau + \int_0^t (u, D_h u)_{1,\Omega} d\tau$$

$$+ \int_0^t \int_\Omega g_1(u_t) D_h u \, dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma D_h u \, d\Gamma d\tau$$

(3.23)
$$= \int_0^t \int_\Omega f_1(u, v) D_h u \, dx d\tau + \int_0^t \int_\Gamma h(\gamma u) \gamma D_h u \, d\Gamma d\tau$$

and

$$(v_t(t), D_h v(t))_\Omega - (v_t(0), D_h v(0))_\Omega - \int_0^t (v_t, (D_h v)_t)_\Omega d\tau + \int_0^t (v, D_h v)_{1,\Omega} d\tau$$

(3.24)
$$+ \int_0^t \int_\Omega g_2(v_t) D_h v \, dx d\tau = \int_0^t \int_\Omega f_2(u, v) D_h v \, dx d\tau.$$

We will pass to the limit as $h \longrightarrow 0$ in (3.23) only, since passing to the limit in (3.24) can be handled in the same way.

Since $u, u_t \in C([0, t]; L^2(\Omega))$, then (3.6) shows

$$D_h u(0) \longrightarrow \frac{1}{2} u_t(0) \text{ and } D_h u(t) \longrightarrow \frac{1}{2} u_t(t) \text{ weakly in } L^2(\Omega).$$

It follows that

(3.25)
$$\lim_{h \longrightarrow 0} (u_t(0), D_h u(0))_\Omega = \frac{1}{2} \|u_t(0)\|_2^2,$$
$$\lim_{h \longrightarrow 0} (u_t(t), D_h u(t))_\Omega = \frac{1}{2} \|u_t(t)\|_2^2.$$

Also, by (3.4)

(3.26)
$$\int_0^t (u_t, (D_h u)_t)_\Omega d\tau = 0.$$

In addition, since $u \in C([0, t]; H^1(\Omega))$, then (3.3) yields

(3.27)
$$\lim_{h \longrightarrow 0} \int_0^t (u, D_h u)_{1,\Omega} d\tau = \frac{1}{2} \left(\|u(t)\|_{1,\Omega}^2 - \|u(0)\|_{1,\Omega}^2 \right).$$

Since $u_t \in L^{m+1}(\Omega \times (0, t))$ and $|g_1(s)| \le \beta |s|^m$ whenever $|s| \ge 1$, then clearly $g_1(u_t) \in L^{\tilde{m}}(\Omega \times (0, t))$, where $\tilde{m} = \frac{m+1}{m}$. Hence, by (3.18)

(3.28)
$$\lim_{h \longrightarrow 0} \int_0^t \int_\Omega g_1(u_t) D_h u \, dx d\tau = \int_0^t \int_\Omega g_1(u_t) u_t \, dx d\tau.$$

Similarly, since $g(\gamma u_t) \in L^{\tilde{q}}(\Gamma \times (0, t))$, then (3.20) implies

$$(3.29) \qquad \lim_{h \to 0} \int_0^t \int_\Gamma g(\gamma u_t) \gamma D_h u \, d\Gamma d\tau = \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t \, dx d\tau.$$

In order to handle the interior source, we note that since $u \in C([0, t]; H^1(\Omega))$ and $v \in C([0, t]; H_0^1(\Omega))$, then there exists $M_0 > 0$ such that $\|u(\tau)\|_6, \|v(\tau)\|_6 \le M_0$ for all $\tau \in [0, t]$. Also, since $|f_1(u, v)| \le C(|u|^p + |v|^p + 1)$, then

$$\int_\Omega |f_1(u(\tau), v(\tau))|^{\frac{6}{p}} dx \le C \int_\Omega (|u(\tau)|^6 + |v(\tau)|^6 + 1) dx \le C(M_0),$$

for all $\tau \in [0, t]$. Hence, $f_1(u, v) \in L^\infty(0, t; L^{\frac{6}{p}}(\Omega))$, and so, $f_1(u, v) \in L^{\frac{6}{p}}(\Omega \times (0, t))$. Since $\frac{6}{p} > \tilde{m}$, then $f_1(u, v) \in L^{\tilde{m}}(\Omega \times (0, t))$. Therefore, it follows from (3.18) that

$$(3.30) \qquad \lim_{h \to 0} \int_0^t \int_\Omega f_1(u, v) D_h u \, dx d\tau = \int_0^t \int_\Omega f_1(u, v) u_t \, dx d\tau.$$

Finally, we consider the boundary source. Again, since $u \in C([0, t]; H^1(\Omega))$ and $H^1(\Omega) \hookrightarrow L^4(\Gamma)$, then there exists $M_1 > 0$ such that $|\gamma u(\tau)|_4 \le M_1$ for all $\tau \in [0, t]$. By recalling the assumption $|h(\gamma u)| \le C(|\gamma u|^k + 1)$, then

$$\int_\Gamma |h(\gamma u(\tau))|^{\frac{4}{k}} dx \le C \int_\Gamma (|\gamma u(\tau)|^4 + 1) d\Gamma \le C(M_1)$$

for all $\tau \in [0, t]$. Hence, $h(\gamma u) \in L^\infty(0, t; L^{\frac{4}{k}}(\Gamma))$, and in particular, $h(\gamma u) \in L^{\frac{4}{k}}(\Gamma \times (0, t))$. Since $\frac{4}{k} > \tilde{q}$, we conclude $h(\gamma u) \in L^{\tilde{q}}(\Gamma \times (0, t))$. Therefore, (3.20) yields

$$(3.31) \qquad \lim_{h \to 0} \int_0^t \int_\Gamma h(\gamma u) \gamma D_h u \, d\Gamma d\tau = \int_0^t \int_\Gamma h(\gamma u) \gamma u_t \, d\Gamma d\tau.$$

By combining (3.25)-(3.31), we can pass to the limit as $h \to 0$ in (3.23) to obtain

$$(3.32)$$
$$\frac{1}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2) + \int_0^t \int_\Omega g_1(u_t) u_t \, dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t \, d\Gamma d\tau$$
$$= \frac{1}{2}(\|u_t(0)\|_2^2 + \|u(0)\|_{1,\Omega}^2) + \int_0^t \int_\Omega f_1(u, v) u_t \, dx d\tau + \int_0^t \int_\Gamma h(\gamma u) \gamma u_t \, d\Gamma d\tau.$$

Similarly, we can also pass to the limit as $h \to 0$ in (3.24) to obtain

$$\frac{1}{2}(\|v_t(t)\|_2^2 + \|v(t)\|_{1,\Omega}^2) + \int_0^t \int_\Omega g_2(v_t) v_t \, dx d\tau$$
$$(3.33) \qquad = \frac{1}{2}(\|v_t(0)\|_2^2 + \|v(0)\|_{1,\Omega}^2) + \int_0^t \int_\Omega f_2(u, v) v_t \, dx d\tau.$$

By adding (3.32) to (3.33), then the energy identity (1.8) follows.

4. Uniqueness of weak solutions

The uniqueness results of Theorem 1.5 and Theorem 1.7 will be justified in the following two subsections.

4.1. **Proof of Theorem 1.5.** The proof of Theorem 1.5 will be carried out in the following four steps.

Step 1. Let (u,v) and $(\hat{u},\hat{v})$ be two weak solutions on $[0,T]$ in the sense of Definition 1.2 satisfying the same initial conditions. Put $y = u - \hat{u}$ and $z = v - \hat{v}$. The energy corresponding to (y,z) is given by

$$(4.1) \qquad \tilde{E}(t) = \frac{1}{2}(\|y(t)\|_{1,\Omega}^2 + \|z(t)\|_{1,\Omega}^2 + \|y_t(t)\|_2^2 + \|z_t(t)\|_2^2)$$

for all $t \in [0,T]$. We aim to show that $\tilde{E}(t) = 0$, and thus $y(t) = 0$ and $z(t) = 0$ for all $t \in [0,T]$.

By the regularity imposed on weak solutions in Definition 1.2, there exists a constant $R > 0$ such that

$$(4.2) \quad \begin{cases} \|u(t)\|_{1,\Omega}, \|\hat{u}(t)\|_{1,\Omega}, \|v(t)\|_{1,\Omega}, \|\hat{v}(t)\|_{1,\Omega} \le R, \\ \|u_t(t)\|_2, \|\hat{u}_t(t)\|_2, \|v_t(t)\|_2, \|\hat{v}_t(t)\|_2 \le R, \\ \int_0^T \|u_t\|_{m+1}^{m+1}\,dt, \int_0^T \|\hat{u}_t\|_{m+1}^{m+1}\,dt, \int_0^T |\gamma u_t|_{q+1}^{q+1}dt, \int_0^T |\gamma \hat{u}_t|_{q+1}^{q+1}dt \le R, \\ \int_0^T \|v_t\|_{r+1}^{r+1}\,dt, \int_0^T \|\hat{v}_t\|_{r+1}^{r+1}\,dt \le R \end{cases}$$

for all $t \in [0,T]$. Since $y(0) = y_t(0) = z(0) = z_t(0) = 0$, then by Definition 1.2, y and z satisfy

$$(y_t(t), \phi(t))_\Omega - \int_0^t (y_t, \phi_t)_\Omega d\tau + \int_0^t (y, \phi)_{1,\Omega} d\tau$$

$$+ \int_0^t \int_\Omega (g_1(u_t) - g_1(\hat{u}_t))\phi\, dx d\tau + \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma\phi\, d\Gamma d\tau$$

$$(4.3) \qquad = \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))\phi\, dx d\tau + \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma\phi\, d\Gamma d\tau$$

and

$$(z_t(t), \psi(t))_\Omega - \int_0^t (z_t, \psi_t)_\Omega d\tau + \int_0^t (z, \psi)_{1,\Omega} d\tau + \int_0^t \int_\Omega (g_2(v_t) - g_2(\hat{v}_t))\psi\, dx d\tau$$

$$(4.4) \qquad = \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))\psi\, dx d\tau,$$

for all $t \in [0,T]$ and for all test functions ϕ and ψ as described in Definition 1.2.

Let $\phi(\tau) = D_h y(\tau)$ in (4.3) and $\psi(\tau) = D_h z(\tau)$ in (4.4) for $\tau \in [0,t]$, where the difference quotients $D_h y$ and $D_h z$ are defined in (3.1). Using a similar argument as in obtaining the energy identity (1.8), we can pass to the limit as $h \longrightarrow 0$ and deduce

$$\frac{1}{2}(\|y(t)\|_{1,\Omega}^2 + \|y_t(t)\|_2^2) + \int_0^t \int_\Omega (g_1(u_t) - g_1(\hat{u}_t))y_t\, dx d\tau$$

$$+ \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t\, d\Gamma d\tau$$

$$(4.5) \qquad = \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t\, dx d\tau + \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t\, d\Gamma d\tau$$

and

$$\frac{1}{2}(\|z(t)\|_{1,\Omega}^2 + \|z_t(t)\|_2^2) + \int_0^t \int_\Omega (g_2(v_t) - g_2(\hat{v}_t))z_t dx d\tau$$

(4.6)
$$= \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau.$$

Adding (4.5) and (4.6) and employing the monotonicity properties of g_1, g_2 yield

$$\tilde{E}(t) \le \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau$$

(4.7)
$$+ \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t d\Gamma d\tau - \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma,$$

for all $t \in [0,T]$ where $\tilde{E}(t)$ is defined in (4.1).

We will estimate each term on the right hand side of (4.7).

Step 2. "Estimate for the terms due to the interior sources".

Put

(4.8) $\quad R_f = \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau.$

First we note that, if $1 \le p \le 3$, then by Remark 2.5 we know f_1 and f_2 are both locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$. In this case, the estimate for R_f is straightforward. By using Hölder's inequality, we have

(4.9)
$$\int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau$$

$$\le \left(\int_0^t \int_\Omega |f_1(u,v) - f_1(\hat{u},\hat{v})|^2 dx d\tau\right)^{1/2} \left(\int_0^t \int_\Omega |y_t|^2 dx d\tau\right)^{1/2}$$

$$\le C(R)\left(\int_0^t (\|y\|_{1,\Omega}^2 + \|z\|_{1,\Omega}^2) d\tau\right)^{1/2} \left(\int_0^t \|y_t\|_2^2 d\tau\right)^{1/2} \le C(R)\int_0^t \tilde{E}(\tau)d\tau.$$

Likewise, $\int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau \le C(R)\int_0^t \tilde{E}(\tau)d\tau$. Therefore, for $1 \le p \le 3$, we have the following estimate for R_f:

(4.10)
$$R_f \le C(R)\int_0^t \tilde{E}(\tau)d\tau.$$

For the case $3 < p < 6$, f_1 and f_2 are not locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$, and therefore the computation in (4.9) does not work. To overcome this difficulty, we shall use a clever idea by Bociu and Lasiecka [8,9] which involves integration by parts. In order to do so, we require f_1 and f_2 to be C^2-functions. More precisely, we impose the following assumption: there exists $F \in C^3(\mathbb{R}^2)$ such that $f_1(u,v) = \partial_u F(u,v)$, $f_2(u,v) = \partial_v F(u,v)$ and $|D^\alpha F(u,v)| \le C(|u|^{p-2} + |v|^{p-2} + 1)$ for all α such that $|\alpha| = 3$. It follows from this assumption that

$f_j \in C^2(\mathbb{R}^2)$, $j = 1, 2$, and

(4.11)
$$\begin{cases} \left| D^\beta f_j(u, v) \right| \leq C(|u|^{p-2} + |v|^{p-2} + 1), & \text{for all } |\beta| = 2, \\ |\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1) \text{ and } |f_j(u, v)| \leq C(|u|^p + |v|^p + 1), \\ |\nabla f_j(u, v) - \nabla f_j(\hat{u}, \hat{v})| \leq C(|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2} + 1)(|y| + |z|), \\ |f_j(u, v) - f_j(\hat{u}, \hat{v})| \leq C(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1)(|y| + |z|), \end{cases}$$

where $y = u - \hat{u}$ and $z = v - \hat{v}$.

Now, we evaluate R_f in the case $3 < p < 6$. By integration by parts in time and by recalling $y(0) = 0$, one has

$$\int_0^t \int_\Omega [f_1(u, v) - f_1(\hat{u}, \hat{v})] y_t \, dx \, d\tau = \int_\Omega [f_1(u(t), v(t)) - f_1(\hat{u}(t), \hat{v}(t))] y(t) \, dx$$

$$- \int_\Omega \int_0^t \left[\nabla f_1(u, v) \cdot \begin{pmatrix} u_t \\ v_t \end{pmatrix} - \nabla f_1(\hat{u}, \hat{v}) \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} \right] y \, d\tau \, dx$$

$$= \int_\Omega [f_1(u(t), v(t)) - f_1(\hat{u}(t), \hat{v}(t))] y(t) \, dx - \int_\Omega \int_0^t \nabla f_1(u, v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} y \, d\tau \, dx$$

(4.12) $$- \int_\Omega \int_0^t [\nabla f_1(u, v) - \nabla f_1(\hat{u}, \hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} y \, d\tau \, dx.$$

As in (4.12), we have a similar expression for $\int_0^t \int_\Omega [f_2(u, v) - f_2(\hat{u}, \hat{v})] z_t \, dx \, d\tau$. Therefore, we deduce

(4.13) $$R_f = P_1 + P_2 + P_3 + P_4 + P_5,$$

where

$$\begin{cases} P_1 = \int_\Omega [f_1(u(t), v(t)) - f_1(\hat{u}(t), \hat{v}(t))] y(t) \, dx, \\ P_2 = \int_\Omega [f_2(u(t), v(t)) - f_2(\hat{u}(t), \hat{v}(t))] z(t) \, dx, \\ P_3 = \int_\Omega \int_0^t [\nabla f_1(u, v) - \nabla f_1(\hat{u}, \hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} y \, d\tau \, dx, \\ P_4 = \int_\Omega \int_0^t [\nabla f_2(u, v) - \nabla f_2(\hat{u}, \hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} z) \, d\tau \, dx, \\ P_5 = \int_\Omega \int_0^t \left(\nabla f_1(u, v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} y + \nabla f_2(u, v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} z \right) d\tau \, dx. \end{cases}$$

By using (4.11) and Young's inequality, we obtain

$$|P_1 + P_2|$$
(4.14)
$$\leq C \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1} + 1)(y^2(t) + z^2(t)) \, dx,$$

(4.15)
$$|P_3 + P_4|$$
$$\leq C \int_\Omega \int_0^t (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2} + 1)(y^2 + z^2)(|\hat{u}_t| + |\hat{v}_t|) \, d\tau \, dx.$$

As for P_5, we integrate by parts one more time and use the assumptions $f_1(u,v) = \partial_u F(u,v)$ and $f_2(u,v) = \partial_v F(u,v)$. Indeed,

$$P_5 = \int_\Omega \int_0^t \Big(\partial_u f_1(u,v)y_t y + \partial_v f_1(u,v)z_t y\Big)d\tau dx$$

$$+ \int_\Omega \int_0^t \Big(\partial_u f_2(u,v)y_t z + \partial_v f_2(u,v)z_t z\Big)d\tau dx$$

$$= \int_\Omega \int_0^t \Big(\frac{1}{2}\partial_u f_1(u,v)(y^2)_t + \partial_{uv}^2 F(u,v)(yz)_t + \frac{1}{2}\partial_v f_2(u,v)(z^2)_t\Big)d\tau dx$$

$$= \int_\Omega \Big(\frac{1}{2}\partial_u f_1(u(t),v(t))y(t)^2 + \partial_{uv}^2 F(u(t),v(t))y(t)z(t)\Big)dx$$

$$+ \int_\Omega \frac{1}{2}\partial_v f_2(u(t),v(t))z(t)^2 dx$$

$$+ \int_\Omega \int_0^t \Big(\frac{1}{2}\nabla\partial_u f_1(u,v)y^2 + \nabla\partial_{uv}^2 F(u,v)yz\Big)\cdot\begin{pmatrix}u_t\\v_t\end{pmatrix}d\tau dx$$

(4.16) $$+ \int_\Omega \int_0^t \frac{1}{2}\nabla\partial_v f_2(u,v)z^2 \cdot \begin{pmatrix}u_t\\v_t\end{pmatrix}d\tau dx.$$

By employing (4.11) and Young's inequality, we deduce that

$$P_5 \leq C \int_\Omega (|u(t)|^{p-1} + |v(t)|^{p-1} + 1)(|y(t)|^2 + |z(t)|^2)dx$$

(4.17) $$+ C \int_\Omega \int_0^t (|u|^{p-2} + |v|^{p-2} + 1)(y^2 + z^2)(|u_t| + |v_t|)d\tau dx.$$

It follows from (4.14), (4.15), (4.17), and (4.13) that

(4.18)

$$R_f \leq C \int_\Omega (|y(t)|^2 + |z(t)|^2)dx + C \int_0^t \int_\Omega (y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|)dxd\tau$$

$$+ C \int_0^t \int_\Omega (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2})(y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|)dxd\tau$$

$$+ C \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1})(|y(t)|^2 + |z(t)|^2)dx.$$

Now, we estimate the terms on the right hand side of (4.18) as follows.

1. Estimate for

$$I_1 = \int_\Omega (|y(t)|^2 + |z(t)|^2)dx.$$

Since $y, y_t \in C([0,T]; L^2(\Omega))$ and $y(0) = 0$, we obtain

(4.19) $$\int_\Omega |y(t)|^2 dx = \int_\Omega \left|\int_0^t y_t(\tau)d\tau\right|^2 dx \leq t \int_0^t \|y_t(\tau)\|_2^2 d\tau \leq 2T \int_0^t \tilde{E}(\tau)d\tau.$$

Likewise, $\int_\Omega |z(t)|^2 dx \leq 2T \int_0^t \tilde{E}(\tau)d\tau$. Therefore,

(4.20) $$I_1 \leq 4T \int_0^t \tilde{E}(\tau)d\tau.$$

2. Estimate for

$$I_2 = \int_0^t \int_\Omega (y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|)dxd\tau.$$

A typical term in I_2 is estimated as follows. By using Hölder's inequality and the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\int_0^t \int_\Omega y^2|u_t|dxd\tau \leq \int_0^t \|y\|_6^2 \|u_t\|_{3/2}\, d\tau$$

$$(4.21) \qquad\qquad \leq C \int_0^t \|y\|_{1,\Omega}^2 \|u_t\|_2\, d\tau \leq C(R) \int_0^t \tilde{E}(\tau)d\tau,$$

where we have used the fact that $\|u_t(t)\|_2 \leq R$ for all $t \in [0,T]$ (see (4.2)). Therefore,

$$(4.22) \qquad\qquad I_2 \leq C(R) \int_0^t \tilde{E}(\tau)d\tau.$$

3. Estimate for

$$I_3 = \int_0^t \int_\Omega (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2})(y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|)dxd\tau.$$

A typical term in I_3 is estimated as follows. Recall the assumption $p\frac{m+1}{m} < 6$ which implies $\frac{6}{6-p} < m+1$. Thus, by using Hölder's inequality and (4.2), one has

$$\int_0^t \int_\Omega |u|^{p-2}y^2|u_t|dxd\tau \leq \int_0^t \|u\|_6^{p-2} \|y\|_6^2 \|u_t\|_{\frac{6}{6-p}}\, d\tau$$

$$(4.23) \qquad \leq C \int_0^t \|u\|_{1,\Omega}^{p-2} \|y\|_{1,\Omega}^2 \|u_t\|_{m+1}\, d\tau \leq C(R) \int_0^t \tilde{E}(\tau) \|u_t\|_{m+1}\, d\tau.$$

Therefore,

$$(4.24) \qquad I_3 \leq C(R) \int_0^t \tilde{E}(\tau)\Big(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1} \Big)d\tau.$$

4. Estimate for

$$I_4 = \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1})(|y(t)|^2 + |z(t)|^2)dx.$$

Estimating I_4 is quite involved. We focus on the typical term $\int_\Omega |u(t)|^{p-1}|y(t)|^2dx$ in the following two cases for the exponent $p \in (3,6)$.

Case 1: $3 < p < 5$. In this case, we have

$$(4.25) \quad \int_\Omega |u(t)|^{p-1}|y(t)|^2dx \leq \int_\Omega |y(t)|^2dx + \int_{\{x\in\Omega:|u(t)|>1\}} |u(t)|^{p-1}|y(t)|^2dx.$$

The first term on the right hand side of (4.25) has already been estimated in (4.19). For the second term, we notice that if $0 < \epsilon < 5-p$, then $|u(t)|^{p-1} \leq |u(t)|^{4-\epsilon}$, since

$|u(t)| > 1$. Again, by using Hölder's inequality, (4.2), (1.3), and (1.4), it follows that

$$\int_{\{x\in\Omega:|u(t)|>1\}} |u(t)|^{p-1}|y(t)|^2 dx \leq \int_\Omega |u(t)|^{4-\epsilon}|y(t)|^2 dx \leq \|u(t)\|_6^{4-\epsilon} \|y(t)\|_{\frac{6}{1+\epsilon/2}}^2$$

$$\leq C \|u(t)\|_{1,\Omega}^{4-\epsilon} \|y(t)\|_{H^{1-\epsilon/4}(\Omega)}^2$$

$$(4.26) \qquad = C(R)\Big(\epsilon \|y(t)\|_{1,\Omega}^2 + C_\epsilon \|y(t)\|_2^2\Big).$$

By using (4.19) and (4.26), then from (4.25) it follows that

$$(4.26) \qquad \int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R)\left(\epsilon\tilde{E}(t) + C_\epsilon T \int_0^t \tilde{E}(\tau)d\tau\right),$$

in the case $3 < p < 5$ and where $0 < \epsilon < 5 - p$.

Case 2: $5 \leq p < 6$. In this case, the assumption $p\frac{m+1}{m} < 6$ implies $m > 5$. Recall that in Theorem 1.5 we required a higher regularity of initial data u_0, v_0, namely, $u_0, v_0 \in L^{\frac{3}{2}(p-1)}(\Omega)$. By density of $C_0(\Omega)$ in $L^{\frac{3}{2}(p-1)}(\Omega)$, then for any $\epsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that $\|u_0 - \phi\|_{\frac{3}{2}(p-1)} < \epsilon^{\frac{1}{p-1}}$.

Now,

$$\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C\int_\Omega |u(t)-u_0|^{p-1}|y(t)|^2 dx + C\int_\Omega |u_0-\phi|^{p-1}|y(t)|^2 dx$$

$$(4.28) \qquad + C\int_\Omega |\phi|^{p-1}|y(t)|^2 dx.$$

Since $p < \frac{6m}{m+1}$ and $m > 5$, then $\frac{3(p-1)}{2(m+1)} < 1$. So, by using Hölder's inequality and the bound $\int_0^T \|u_t\|_{m+1}^{m+1} dt \leq R$, one has

$$\int_\Omega |u(t)-u_0|^{p-1}|y(t)|^2 dx \leq \left(\int_\Omega |u(t)-u(0)|^{\frac{3(p-1)}{2}} dx\right)^{2/3} \|y(t)\|_6^2$$

$$\leq C\left(\int_\Omega \left|\int_0^t u_t(\tau)d\tau\right|^{\frac{3(p-1)}{2}} dx\right)^{2/3} \|y(t)\|_{1,\Omega}^2$$

$$\leq C\left[\int_\Omega \left(\int_0^t |u_t|^{m+1}d\tau\right)^{\frac{3(p-1)}{2(m+1)}} dx\right]^{2/3} T^{\frac{m(p-1)}{m+1}}\tilde{E}(t)$$

$$(4.29) \qquad \leq C(R)T^{\frac{m(p-1)}{m+1}}\tilde{E}(t),$$

where we have used the important fact that $\frac{3(p-1)}{2(m+1)} < 1$.

The second term on the right hand side of (4.28) is easily estimated as follows:

$$(4.30) \qquad \int_\Omega |u_0-\phi|^{p-1}|y(t)|^2 dx \leq \|u_0-\phi\|_{\frac{3(p-1)}{2}}^{p-1} \|y(t)\|_6^2 \leq C\epsilon\tilde{E}(t).$$

Since $\phi \in C_0(\Omega)$, then $|\phi(x)| \leq C(\epsilon)$, for all $x \in \Omega$. So, by (4.19), the last term on the right hand side of (4.28) is estimated as follows:

$$(4.31) \qquad \int_\Omega |\phi|^{p-1}|y(t)|^2 dx \leq C(\epsilon)\int_\Omega |y(t)|^2 dx \leq C(\epsilon,T)\int_0^t \tilde{E}(\tau)d\tau.$$

By combining (4.29)-(4.31), then (4.28) yields

$$(4.32) \qquad \int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}(t) + C(\epsilon,T)\int_0^t \tilde{E}(\tau)d\tau,$$

in the case $5 \leq p < 6$.

By combining the estimates in (4.27) and (4.32), then for the case $3 < p < 6$, one has

$$(4.33) \quad \int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}(t) + C(\epsilon,R,T)\int_0^t \tilde{E}(\tau)d\tau,$$

where $\epsilon > 0$ such that $\epsilon < 5 - p$, if $3 < p < 5$.

The other terms in I_4 can be estimated in the same way, and we have

$$(4.34) \qquad I_4 \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\right)\tilde{E}(t) + C(\epsilon,R,T)\int_0^t \tilde{E}(\tau)d\tau.$$

Finally, by combining the estimates (4.20), (4.22), (4.24) and (4.34) back into (4.18), we obtain for $3 < p < 6$:

$$R_f \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\right)\tilde{E}(t)$$

$$(4.35) \quad + C(\epsilon,R,T)\int_0^t \tilde{E}(\tau)(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1} + 1)d\tau,$$

where $\epsilon > 0$ is sufficiently small. According to (4.10), estimate (4.35) also holds for $1 \leq p \leq 3$, i.e., (4.35) holds for all $1 \leq p < 6$.

Step 3. Estimate for

$$R_h = \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t d\Gamma d\tau.$$

First, we consider the subcritical case: $1 \leq k < 2$. Although, in this case, h is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$, we cannot estimate R_h by using the same method as we have done for R_f. More precisely, an estimate as in (4.9) won't work for R_h, because the energy $\tilde{E}$ does not control the boundary trace γy_t.

In order to overcome this difficulty, we shall take advantage of the boundary damping term: $\int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma d\tau$. It is here where the strong monotonicity condition imposed on g in Assumption 1.4 is critical; namely, the assumption that there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$. Now, by recalling $y = u - \hat{u}$, we have

$$(4.36) \qquad \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma d\tau \geq m_g \int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau.$$

To estimate R_h, we employ Hölder's inequality followed by Young's inequality, and the fact that h is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$ when $1 \leq k < 2$ (see Remark 2.8). Thus,

$$R_h \leq \left(\int_0^t \int_\Gamma |h(\gamma u) - h(\gamma \hat{u})|^2 d\Gamma d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau \right)^{\frac{1}{2}}$$

$$\leq C(R) \left(\int_0^t \|y\|_{1,\Omega}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau \right)^{\frac{1}{2}}$$

$$(4.37) \qquad \leq \epsilon \int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau + C(R,\epsilon) \int_0^t \tilde{E}(\tau) d\tau.$$

Therefore, if we choose $\epsilon \leq m_g$, then by (4.36) and (4.37), we obtain for $1 \leq k < 2$:

$$(4.38) \qquad R_h - \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t)) \gamma y_t \leq C(R,\epsilon) \int_0^t \tilde{E}(\tau) d\tau.$$

Next, we consider the case $2 \leq k < 4$. In this case, we need the extra assumption $h \in C^2(\mathbb{R})$ such that $h''(s) \leq C(|s|^{k-2} + 1)$, which implies

$$(4.39) \qquad \begin{cases} |h'(s)| \leq C(|s|^{k-1} + 1), \quad |h(s)| \leq C(|s|^k + 1), \\ |h'(u) - h'(\hat{u})| \leq C(|u|^{k-2} + |\hat{u}|^{k-2} + 1)|y|, \\ |h(u) - h(\hat{u})| \leq C(|u|^{k-1} + |\hat{u}|^{k-1} + 1)|y|, \end{cases}$$

where $y = u - \hat{u}$.

To evaluate R_h, integrate by parts twice with respect to time, employ (4.39) and the fact that $y(0) = 0$ to obtain

$$(4.40)$$

$$R_h \leq \left| \int_\Gamma [h(\gamma u(t)) - h(\gamma \hat{u}(t))] \gamma y(t) d\Gamma \right| + \left| \int_0^t \int_\Gamma [h'(\gamma u)\gamma u_t - h'(\gamma \hat{u})\gamma \hat{u}_t] \gamma y d\Gamma d\tau \right|$$

$$\leq \left| \int_\Gamma [h(\gamma u(t)) - h(\gamma \hat{u}(t))] \gamma y(t) d\Gamma \right| + \left| \int_0^t \int_\Gamma [h'(\gamma u) - h'(\gamma \hat{u})] \gamma \hat{u}_t \gamma y d\Gamma d\tau \right|$$

$$+ \frac{1}{2} \left| \int_\Gamma h'(\gamma u(t))(\gamma y(t))^2 d\Gamma \right| + \frac{1}{2} \left| \int_0^t \int_\Gamma h''(\gamma u)\gamma u_t (\gamma y)^2 d\Gamma d\tau \right|$$

$$\leq I_5 + I_6 + I_7 + I_8,$$

where

$$I_5 = C \int_\Gamma |\gamma y(t)|^2 d\Gamma,$$

$$I_6 = C \int_0^t \int_\Gamma (|\gamma u_t| + |\gamma \hat{u}_t|)|\gamma y|^2 d\Gamma d\tau,$$

$$I_7 = C \int_0^t \int_\Gamma (|\gamma u|^{k-2} + |\gamma \hat{u}|^{k-2})(|\gamma u_t| + |\gamma \hat{u}_t|)|\gamma y|^2 d\Gamma d\tau,$$

$$I_8 = C \int_\Gamma (|\gamma u(t)|^{k-1} + |\gamma \hat{u}(t)|^{k-1})|\gamma y(t)|^2 d\Gamma.$$

Since $y(t) \in H^1(\Omega)$, then I_5 is easily estimated as follows:

$$I_5 = |\gamma y(t)|_2^2 \leq C \|y(t)\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \epsilon \|y(t)\|_{1,\Omega}^2 + C_\epsilon \|y(t)\|_2^2$$

$$(4.41) \qquad\qquad\qquad \leq 2\epsilon \tilde{E}(t) + C_\epsilon T \int_0^t \tilde{E}(\tau)d\tau,$$

where we have used (1.3), (1.4) and (4.19).

Since $q \geq 1$ and $H^1(\Omega) \hookrightarrow L^4(\Gamma)$, then I_6 is estimated by

$$(4.42) \quad I_6 \leq C \int_0^t (|\gamma u_t|_2 + |\gamma \hat{u}_t|_2)|\gamma y|_4^2 d\tau \leq C \int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1})\tilde{E}(\tau)d\tau.$$

In I_7 we focus on the typical term $\int_0^t \int_\Gamma |\gamma u|^{k-2}|\gamma u_t||\gamma y|^2 d\Gamma d\tau$. Notice that the assumption $k\frac{q+1}{q} < 4$ implies $\frac{4}{4-k} < q+1$. Therefore,

$$\int_0^t \int_\Gamma |\gamma u|^{k-2}|\gamma u_t||\gamma y|^2 d\Gamma d\tau \leq \int_0^t |\gamma u|_4^{k-2}|\gamma u_t|_{\frac{4}{4-k}}|\gamma y|_4^2 d\tau$$

$$(4.43) \qquad \leq C \int_0^t \|u\|_{1,\Omega}^{k-2} |\gamma u_t|_{q+1} \|y\|_{1,\Omega}^2 d\tau \leq C(R) \int_0^t |\gamma u_t|_{q+1}\tilde{E}(\tau)d\tau,$$

where we have used (4.2). The other terms in I_7 can be estimated in the same manner, thus

$$(4.44) \qquad\qquad I_7 \leq C(R) \int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1})\tilde{E}(\tau)d\tau.$$

Finally, we estimate I_8 by focusing on the typical term $\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma$. We consider the following two cases for the exponent $k \in [2, 4)$.

Case 1: $2 \leq k < 3$. First, we note that

$$(4.45)$$
$$\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \leq \int_\Gamma |\gamma y(t)|^2 d\Gamma + \int_{\{x \in \Gamma : |\gamma u(t)| > 1\}} |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma.$$

The first term on the right hand side of (4.45) has been estimated in (4.41). As for the second term, we choose $0 < \epsilon < 3 - k$, and so, $k - 1 < 2 - \epsilon$. By using Hölder's inequality, (1.3) and (1.4), we obtain

$$\int_{\{x \in \Gamma : |\gamma u(t)| > 1\}} |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \leq \int_\Gamma |\gamma u(t)|^{2-\epsilon}|\gamma y(t)|^2 d\Gamma$$

$$\leq |\gamma u(t)|_4^{2-\epsilon}|\gamma y(t)|_{\frac{4}{1+\epsilon/2}}^2$$

$$\leq C \|u(t)\|_{1,\Omega}^{2-\epsilon} \|y(t)\|_{H^{1-\epsilon/4}(\Omega)}^2$$

$$(4.46) \qquad\qquad \leq C(R)(\epsilon \|y(t)\|_{1,\Omega}^2 + C_\epsilon \|y(t)\|_2^2).$$

Therefore, by using the estimates (4.46), (4.41) and (4.19), we obtain from (4.45) that

$$(4.47) \qquad \int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \leq C(R)\left(\epsilon \tilde{E}(t) + C_\epsilon T \int_0^t \tilde{E}(\tau)d\tau\right)$$

for the case $2 < k < 3$, and where $0 < \epsilon < 3 - k$.

Case 2: $3 \leq k < 4$. Observe that, in this case, the assumption $k\frac{q+1}{q} < 4$ implies $q > 3$. Also, recall that in Theorem 1.5 we required the extra assumption: $\gamma u_0 \in L^{2(k-1)}(\Gamma)$.

By the density of $C(\Gamma)$ in $L^{2(k-1)}(\Gamma)$, for any $\epsilon > 0$, there exists $\psi \in C(\Gamma)$ such that $|\gamma u_0 - \psi|_{2(k-1)} \leq \epsilon^{\frac{1}{k-1}}$. Note that

$$\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \leq C \int_\Gamma |\gamma u(t) - \gamma u_0|^{k-1}|\gamma y(t)|^2 d\Gamma$$

$$(4.48) \qquad + C \int_\Gamma |\gamma u_0 - \psi|^{k-1}|\gamma y(t)|^2 d\Gamma + C \int_\Gamma |\psi|^{k-1}|\gamma y(t)|^2 d\Gamma.$$

Since $k < \frac{4q}{q+1}$ and $q > 3$, then $\frac{2(k-1)}{q+1} < 1$. Therefore, by using (4.2), we have

$$(4.49)$$

$$\int_\Gamma |\gamma u(t) - \gamma u_0|^{k-1}|\gamma y(t)|^2 d\Gamma \leq \left(\int_\Gamma \left|\int_0^t \gamma u_t(\tau)d\tau\right|^{2(k-1)} d\Gamma\right)^{\frac{1}{2}} \left(\int_\Gamma |\gamma y(t)|^4 d\Gamma\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_\Gamma \left|\int_0^t |\gamma u_t(\tau)|^{q+1}d\tau\right|^{\frac{2(k-1)}{q+1}} d\Gamma\right)^{\frac{1}{2}} T^{\frac{q(k-1)}{q+1}} \|y(t)\|_{1,\Omega}^2 \leq C(R)T^{\frac{q(k-1)}{q+1}}\tilde{E}(t).$$

The second term on the right hand side of (4.48) is estimated by

$$(4.50)$$
$$\int_\Gamma |\gamma u_0 - \psi|^{k-1}|\gamma y(t)|^2 d\Gamma \leq |\gamma u_0 - \psi|_{2(k-1)}^{k-1}|\gamma y(t)|_4^2 \leq C\epsilon \|y(t)\|_{1,\Omega}^2 \leq C\epsilon\tilde{E}(t).$$

Finally, we estimate the last term on the right hand side of (4.48). Since $\psi \in C(\Gamma)$, then $|\psi(x)| \leq C(\epsilon)$, for all $x \in \Gamma$. It follows from (4.41) that

$$\int_\Gamma |\psi|^{k-1}|\gamma y(t)|^2 d\Gamma \leq C(\epsilon) \int_\Gamma |\gamma y(t)|^2 d\Gamma$$

$$(4.51) \qquad\qquad\qquad \leq \epsilon C(\epsilon)\tilde{E}(t) + C(\epsilon, T) \int_0^t \tilde{E}(\tau)d\tau.$$

Now, (4.49)-(4.51) and (4.48) yield

$$(4.52) \quad \int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon)\tilde{E}(t) + C(\epsilon, T) \int_0^t \tilde{E}(\tau)d\tau,$$

for the case $3 \leq k < 4$. It is easy to see that the other term in I_8 has the same estimate as (4.47) and (4.52). So, we may conclude that for $2 < k < 4$ and sufficiently small $\epsilon > 0$:

$$(4.53) \qquad I_8 \leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon)\tilde{E}(t) + C(\epsilon, R, T) \int_0^t \tilde{E}(\tau)d\tau.$$

Combine (4.41), (4.42), (4.44) and (4.53) back into (4.40) to obtain the following estimate for R_h in the case $2 \leq k < 4$:

$$R_h \leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon)\tilde{E}(t)$$

$$(4.54) \qquad + C(\epsilon, R, T) \int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1} + 1)\tilde{E}(\tau)d\tau,$$

where $\epsilon > 0$ is sufficiently small.

Step 4. Completion of the proof.

By the estimates (4.35), (4.38), (4.54) and employing the monotonicity property of g, we obtain from (4.7) that

$$\tilde{E}(t) \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right)\tilde{E}(t)$$

$$+ C(\epsilon, R, T)\int_0^t \tilde{E}(\tau)\Big(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1}$$

$$+ |\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1} + 1\Big)d\tau,$$

for all $t \in [0,T]$. Choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right) < 1.$$

By applying Gronwall's inequality with an L^1-kernel, it follows that $\tilde{E}(t) = 0$ on $[0,T]$. Hence, $y(t) = z(t) = 0$ on $[0,T]$. Finally we note that it is sufficient to consider a small time interval $[0,T]$, since this process can be reiterated. The proof of Theorem 1.5 is now complete.

4.2. **Proof of Theorem 1.7.** We begin by pointing out that the only difference between Theorem 1.7 and Theorem 1.5 is that Assumption 1.4 (a) is not imposed in Theorem 1.7. Thus, the proof of Theorem 1.7 is essentially the same as Theorem 1.5, with the exception of the estimate for R_f in (4.8). So, we focus on estimating R_f in the case $p > 3$ and the interior sources f_1, f_2 are not necessarily C^2-functions. With this scenario in place, the method of integration by parts fails twice. To handle this difficulty, recall the additional restriction on parameters and the initial data in Theorem 1.7, namely, $m, r \geq 3p - 4$, if $p > 3$, and $u_0, v_0 \in L^{3(p-1)}(\Omega)$.

Now, since $|\nabla f_1(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, then by the mean value theorem,

$$(4.55) \quad |f_1(u,v) - f_1(\hat{u},\hat{v})| \leq C(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1)(|y| + |z|),$$

where $y = u - \hat{u}$ and $z = v - \hat{v}$. Thus,

$$(4.56) \qquad \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau \leq I_1 + I_2,$$

where

$$I_1 = C\int_0^t \int_\Omega (|y| + |z|)|y_t| dx d\tau,$$

$$I_2 = C\int_0^t \int_\Omega (|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1})(|y| + |z|)|y_t| dx d\tau.$$

The estimate for I_1 is straightforward. Invoking Hölder's inequality yields

$$(4.57) \quad I_1 \leq C\int_0^t (\|y\|_6 + \|z\|_6)\|y_t\|_2 \, d\tau \leq C\int_0^t \tilde{E}(\tau)^{\frac{1}{2}}\tilde{E}(\tau)^{\frac{1}{2}}d\tau = C\int_0^t \tilde{E}(\tau)d\tau.$$

A typical term in I_2 is estimated as follows:

$$\int_0^t \int_\Omega |u|^{p-1}|y||y_t| dx d\tau$$

$$(4.58) \qquad \leq C\int_0^t \int_\Omega |u-u_0|^{p-1}|y||y_t| dx d\tau + C\int_0^t \int_\Omega |u_0|^{p-1}|y||y_t| dx d\tau.$$

By Hölder's inequality,

$$\int_0^t \int_\Omega |u - u_0|^{p-1} |y| |y_t| dx d\tau$$

$$(4.59) \quad \leq \int_0^t \left(\int_\Omega |u(\tau) - u_0|^{3(p-1)} dx \right)^{\frac{1}{3}} \left(\int_\Omega |y(\tau)|^6 dx \right)^{\frac{1}{6}} \left(\int_\Omega |y_t(\tau)|^2 dx \right)^{\frac{1}{2}} d\tau.$$

Since $u, u_t \in C([0,T]; L^2(\Omega))$, we can write

$$\int_\Omega |u(\tau) - u_0|^{3(p-1)} dx = \int_\Omega \left| \int_0^\tau u_t(s) ds \right|^{3(p-1)} dx$$

$$(4.60) \qquad \leq C(T) \int_\Omega \left(\int_0^\tau |u_t(s)|^{m+1} ds \right)^{\frac{3(p-1)}{m+1}} dx.$$

Since $m \geq 3p - 4$, then $\frac{3(p-1)}{m+1} \leq 1$. Therefore, by using Hölder's inequality and (4.2), it follows that

$$(4.61) \quad \int_\Omega |u(\tau) - u_0|^{3(p-1)} dx \leq C(T) \left(\int_\Omega \int_0^\tau |u_t(s)|^{m+1} ds dx \right)^{\frac{3(p-1)}{m+1}} \leq C(R,T).$$

So, (4.61) and (4.59) yield

$$\int_0^t \int_\Omega |u - u_0|^{p-1} |y| |y_t| dx d\tau \leq C(R,T) \int_0^t \|y(\tau)\|_6 \|y_t(\tau)\|_2 d\tau$$

$$(4.62) \qquad \leq C(R,T) \int_0^t \tilde{E}(\tau)^{\frac{1}{2}} \tilde{E}(\tau)^{\frac{1}{2}} d\tau = C(R,T) \int_0^t \tilde{E}(\tau) d\tau.$$

By recalling the assumption $u_0 \in L^{3(p-1)}(\Omega)$, then the second term on the right hand side of (4.58) is estimated by

$$\int_0^t \int_\Omega |u_0|^{p-1} |y| |y_t| dx d\tau \leq \int_0^t \|u_0\|_{3(p-1)}^{p-1} \|y(\tau)\|_6 \|y_t(\tau)\|_2 d\tau$$

$$(4.63) \qquad\qquad\qquad \leq C \|u_0\|_{3(p-1)}^{p-1} \int_0^t \tilde{E}(\tau) d\tau.$$

Combining (4.62) and (4.63) back into (4.58) yields

$$(4.64) \qquad \int_0^t \int_\Omega |u|^{p-1} |y| |y_t| dx d\tau \leq C\left(R,T, \|u_0\|_{3(p-1)}\right) \int_0^t \tilde{E}(\tau) d\tau.$$

The other terms in I_2 are estimated in the same manner, and one has

$$(4.65) \qquad I_2 \leq C\left(R,T, \|u_0\|_{3(p-1)}, \|v_0\|_{3(p-1)}\right) \int_0^t \tilde{E}(\tau) d\tau.$$

Hence, (4.57), (4.65), and (4.56) yield

$$\int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u}, \hat{v})) y_t dx d\tau$$

$$(4.66) \qquad\qquad \leq C\left(R,T, \|u_0\|_{3(p-1)}, \|v_0\|_{3(p-1)}\right) \int_0^t \tilde{E}(\tau) d\tau.$$

It is clear that $\int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u}, \hat{v})) z_t dx d\tau$ has the same estimate as in (4.66). Finally, we may use the same argument as in Step 3 and Step 4 in the proof of Theorem 1.5 and complete the proof of Theorem 1.7.

5. Global existence

This section is devoted to proving the existence of global solutions (Theorem 1.8). Here, we apply a standard continuation procedure for ODE's to conclude that either the weak solution (u, v) is global or there exists $0 < T < \infty$ such that $\limsup_{t \longrightarrow T^-} E_1(t) = +\infty$, where $E_1(t)$ is the modified energy defined by

$$E_1(t) := \frac{1}{2}(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2)$$

$$(5.1) \qquad + \frac{1}{p+1}(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}) + \frac{1}{k+1}|\gamma u(t)|_{k+1}^{k+1}.$$

We aim to show that the latter cannot happen under the assumptions of Theorem 1.8. Indeed, this assertion is contained in the following proposition.

Proposition 5.1. *Let (u, v) be a weak solution of* (1.1) *on* $[0, T_0]$ *as furnished by Theorem 1.3. Assume u_0, $v_0 \in L^{p+1}(\Omega)$, if $p > 5$, and $\gamma u_0 \in L^{k+1}(\Gamma)$, if $k > 3$. We have:*

- *If $p \le \min\{m, r\}$ and $k \le q$, then for all $t \in [0, T_0]$, (u, v) satisfies*

$$(5.2) \qquad E_1(t) + \int_0^t \left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1} \right) d\tau \le C(T_0, E_1(0)),$$

  where $T_0 > 0$ is being arbitrary.
- *If $p > \min\{m, r\}$ or $k > q$, then the bound in* (5.2) *holds for $0 \le t < T'$, for some $T' > 0$ depending on $E_1(0)$ and T_0.*

Proof. With the modified energy as given in (5.1), the energy identity (1.8) yields

$$E_1(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t] \, dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau$$

$$= E_1(0) + \int_0^t \int_\Omega [f_1(u, v)u_t + f_2(u, v)v_t] \, dx d\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau$$

$$+ \frac{1}{p+1} \int_\Omega (|u(t)|^{p+1} - |u(0)|^{p+1} + |v(t)|^{p+1} - |v(0)|^{p+1}) dx$$

$$+ \frac{1}{k+1} \int_\Gamma (|\gamma u(t)|^{k+1} - |\gamma u(0)|^{k+1}) d\Gamma$$

$$= E_1(0) + \int_0^t \int_\Omega [f_1(u, v)u_t + f_2(u, v)v_t] \, dx d\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau$$

$$(5.3) \qquad + \int_0^t \int_\Omega (|u|^{p-1}uu_t + |v|^{p-1}vv_t) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^{k-1}\gamma u\gamma u_t d\Gamma d\tau.$$

To estimate the source terms on the right hand side of (5.3), we recall the assumptions $|h(s)| \le C(|s|^k + 1)$, $|f_j(u, v)| \le C(|u|^p + |v|^p + 1)$, $j = 1, 2$. So, by employing

Hölder's and Young's inequalities, we find

$$\left| \int_0^t \int_\Omega f_1(u,v) u_t \, dx d\tau \right| \leq C \int_0^t \int_\Omega (|u|^p + |v|^p + 1)|u_t| \, dx d\tau$$

$$\leq C \int_0^t \|u_t\|_{p+1} \left(\|u\|_{p+1}^p + \|v\|_{p+1}^p + |\Omega|^{\frac{p}{p+1}} \right) d\tau$$

$$\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} \, d\tau + C_\epsilon \int_0^t \left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\Omega| \right) d\tau$$

$$(5.4) \qquad \leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} \, d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Omega|.$$

Similarly, we deduce

$$(5.5) \qquad \left| \int_0^t \int_\Omega f_2(u,v) v_t \, dx d\tau \right| \leq \epsilon \int_0^t \|v_t\|_{p+1}^{p+1} \, d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Omega|$$

and

$$(5.6) \qquad \left| \int_0^t \int_\Gamma h(\gamma u) \gamma u_t \right| \leq \epsilon \int_0^t |\gamma u_t|_{k+1}^{k+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Gamma|.$$

By adopting similar estimates as in (5.4), we obtain

$$\left| \int_0^t \int_\Omega (|u|^{p-1} u u_t + |v|^{p-1} v v_t) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^{k-1} \gamma u \gamma u_t d\Gamma d\tau \right|$$

$$\leq \int_0^t \int_\Omega (|u|^p |u_t| + |v|^p |v_t|) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^k |\gamma u_t| d\Gamma d\tau$$

$$(5.7) \qquad \leq \epsilon \int_0^t (\|u_t\|_{p+1}^{p+1} + \|v_t\|_{p+1}^{p+1} + |\gamma u_t|_{k+1}^{k+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau.$$

By recalling (2.34), one has

$$\int_0^t \int_\Omega [g_1(u_t) u_t + g_2(v_t) v_t] \, dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t d\Gamma d\tau$$

$$(5.8) \qquad \geq \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau - \alpha T_0(2|\Omega| + |\Gamma|).$$

Now, if $p \leq \min\{m, r\}$ and $k \leq q$, it follows from (5.4)-(5.8) and the energy identity (5.3) that, for $t \in [0, T_0]$,

$$(5.9)$$

$$E_1(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau$$

$$\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{p+1}^{p+1} + \|v_t\|_{p+1}^{p+1} + |\gamma u_t|_{k+1}^{k+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_{T_0, \epsilon}$$

$$\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_{T_0, \epsilon},$$

where we have used Hölder's and Young's inequalities in the last line of (5.9). By choosing $0 < \epsilon \leq \alpha/2$, then (5.9) yields

$$E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau$$

$$(5.10) \qquad\qquad \leq C_\epsilon \int_0^t E_1(\tau) d\tau + E_1(0) + C_{T_0,\epsilon}.$$

In particular,

$$(5.11) \qquad\qquad E_1(t) \leq C_\epsilon \int_0^t E_1(\tau) d\tau + E_1(0) + C_{T_0,\epsilon}.$$

By Gronwall's inequality, we conclude that

$$(5.12) \qquad\qquad E_1(t) \leq (E_1(0) + C_{T_0,\epsilon}) e^{C_\epsilon T_0} \text{ for } t \in [0, T_0],$$

where $T_0 > 0$ is arbitrary, and by combining (5.10) and (5.12), the desired result in (5.2) follows.

Now, if $p > \min\{m, r\}$ or $k > q$, then we slightly modify estimate (5.4) by using different Hölder's conjugates. Specifically, we apply Hölder's inequality with $m + 1$ and $\tilde{m} = \frac{m+1}{m}$ followed by Young's inequality to obtain

$$\left| \int_0^t \int_\Omega f_1(u, v) u_t dx d\tau \right| \leq C \int_0^t \int_\Omega (|u|^p + |v|^p + 1) |u_t| dx d\tau$$

$$\leq C \int_0^t \|u_t\|_{m+1} \left(\|u\|_{p\tilde{m}}^p + \|v\|_{p\tilde{m}}^p + |\Omega|^{1/\tilde{m}} \right) d\tau$$

$$(5.13) \qquad \leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p\tilde{m}}^{p\tilde{m}} + \|v\|_{p\tilde{m}}^{p\tilde{m}} + |\Omega| \right) d\tau.$$

Since $p\tilde{m} < 6$ and $H^1(\Omega) \hookrightarrow \mathrm{L}^6(\Omega)$, then

$$\left| \int_0^t \int_\Omega f_1(u, v) u_t dx d\tau \right| \leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{1,\Omega}^{p\tilde{m}} + \|v\|_{1,\Omega}^{p\tilde{m}} + |\Omega| \right) d\tau$$

$$(5.14) \qquad\qquad \leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}} d\tau + C_\epsilon T_0 |\Omega|.$$

Likewise, we may deduce

$$(5.15) \quad \left| \int_0^t \int_\Omega f_2(u, v) v_t dx d\tau \right| \leq \epsilon \int_0^t \|v_t\|_{r+1}^{r+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{r}}{2}} d\tau + C_\epsilon T_0 |\Omega|$$

and

$$(5.16) \quad \left| \int_0^t \int_\Gamma h(\gamma u) \gamma u_t \right| \leq \epsilon \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{k\tilde{q}}{2}} d\tau + C_\epsilon T_0 |\Gamma|.$$

In addition, by employing similar estimates as in (5.13)-(5.14), we have

$$\left| \int_0^t \int_\Omega (|u|^{p-1}uu_t + |v|^{p-1}vv_t)dxd\tau + \int_0^t \int_\Gamma |\gamma u|^{k-1}\gamma u\gamma u_t d\Gamma d\tau \right|$$

$$\leq \int_0^t \int_\Omega (|u|^p|u_t| + |v|^p|v_t|)dxd\tau + \int_0^t \int_\Gamma |\gamma u|^k|\gamma u_t|d\Gamma d\tau$$

$$\leq \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau$$

(5.17)
$$+ C_\epsilon \int_0^t (E_1(\tau)^{\frac{p\tilde{m}}{2}} + E_1(\tau)^{\frac{p\tilde{r}}{2}} + E_1(\tau)^{\frac{k\tilde{q}}{2}})d\tau.$$

By using (5.14)-(5.17) along with (5.8), we obtain from the energy identity (5.3) that

(5.18)

$$E_1(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau$$

$$\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau + C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon},$$

where $\sigma = \max\{\frac{p\tilde{m}}{2}, \frac{p\tilde{r}}{2}, \frac{k\tilde{q}}{2}\} > 1$. Notice that the assumption $p > \min\{m, r\}$ or $k > q$ implies that $\sigma > 1$. By choosing $0 < \epsilon \leq \alpha/2$, then it follows that

$$E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau$$

(5.19)
$$\leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon} \quad \text{for } t \in [0, T_0].$$

In particular,

(5.20)
$$E_1(t) \leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon} \quad \text{for } t \in [0, T_0].$$

By using a standard comparison theorem (see [22] for instance), then (5.20) yields that $E_1(t) \leq z(t)$, where $z(t) = [(E_1(0)+C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma-1)t]^{-\frac{1}{\sigma-1}}$ is the solution of the Volterra integral equation

$$z(t) = C_\epsilon \int_0^t z(s)^\sigma ds + E_1(0) + C_{T_0,\epsilon}.$$

Since $\sigma > 1$, then clearly $z(t)$ blows up at the finite time $T_1 = \frac{1}{C_\epsilon(\sigma-1)}(E_1(0) + C_{T_0,\epsilon})^{1-\sigma}$, i.e., $z(t) \longrightarrow \infty$, as $t \longrightarrow T_1^-$. Note that T_1 depends on the initial energy $E_1(0)$ and the original existence time T_0. Nonetheless, if we choose $T' = \min\{T_0, \frac{1}{2}T_1\}$, then

(5.21)
$$E_1(t) \leq z(t) \leq C_0 := [(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma-1)T']^{-\frac{1}{\sigma-1}},$$

for all $t \in [0, T']$. Finally, we may combine (5.19) and (5.21) to obtain

(5.22)

$$E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau \leq C_\epsilon T' C_0^\sigma + E_1(0) + C_{T_0,\epsilon},$$

for all $t \in [0, T']$, which completes the proof of the proposition. $\square$

6. CONTINUOUS DEPENDENCE ON INITIAL DATA

In this section, we provide the proof of Theorem 1.10. The strategy here is to adopt the same argument as in the proof of Theorem 1.5 and use the bounds of Proposition 5.1.

Proof. Let $U_0 = (u_0, v_0, u_1, v_1) \in X$, where

$$X = \left(H^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)\right) \times \left(H^1_0(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)\right) \times L^2(\Omega) \times L^2(\Omega)$$

such that $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. Assume that $\{U_0^n = (u_0^n, u_1^n, v_0^n, v_1^n)\}$ is a sequence of initial data that satisfies

(6.1) $U_0^n \longrightarrow U_0$ in X and $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{2(k-1)}(\Gamma)$, as $n \longrightarrow \infty$.

Notice that in Remark 1.11, we have pointed out that if $p \leq 5$, then the space X is identical to $H = H^1(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, and if $k \leq 3$, then the assumption $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{2(k-1)}(\Gamma)$ is redundant.

Let $\{(u^n, v^n)\}$ and (u, v) be the unique weak solutions to (1.1) defined on $[0, T_0]$ in the sense of Definition 1.2, corresponding to the initial data $\{U_0^n\}$ and $\{U_0\}$, respectively. First, we show that the local existence time T_0 can be taken independent of $n \in \mathbb{N}$. To see this, we recall that the local existence time provided by Theorem 1.3 depends on the initial energy $E(0)$. In addition, since $U_0^n \longrightarrow U_0$ in X, then $u_0^n \longrightarrow u_0$, $v_0^n \longrightarrow v_0$ in $L^{p+1}(\Omega)$, if $p > 5$, and $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{k+1}(\Gamma)$, if $k > 3$. Hence, we may assume $E_1^n(0) \leq E_1(0) + 1$, for all $n \in \mathbb{N}$, where $E_1(t)$ is defined in (5.1) and $E_1^n(t)$ is defined by

$$E_1^n(t) := E^n(t) + \frac{1}{p+1}(\|u^n(t)\|_{p+1}^{p+1} + \|v^n(t)\|_{p+1}^{p+1}) + \frac{1}{k+1}|\gamma u^n(t)|_{k+1}^{k+1},$$

where $E^n(t) = \frac{1}{2}(\|u^n(t)\|_{1,\Omega}^2 + \|v^n(t)\|_{1,\Omega}^2 + \|u_t^n(t)\|_2^2 + \|v_t^n(t)\|_2^2)$. Therefore, we can choose K, as in (2.38), sufficiently large, say $K^2 \geq 4E_1(0) + 5$. Then the local existence time T_0 for the solutions $\{(u^n, v^n)\}$ and (u, v) can be chosen independent of $n \in \mathbb{N}$. Moreover, in view of (5.2), T_0 can be taken arbitrarily large in the case when $p \leq \min\{m, r\}$ and $k \leq q$. However, in the case when $p > \min\{m, r\}$ or $k > q$, we select the local existence time to be $T = T'$, where T' is given in Proposition 5.1 (which is also uniform in n). In either case, it follows from (5.2) that there exists $R > 0$ such that, for all $n \in \mathbb{N}$ and all $t \in [0, T]$,

(6.2) $$\begin{cases} E_1(t) + \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1})d\tau \leq R, \\ E_1^n(t) + \int_0^t (\|u_t^n\|_{m+1}^{m+1} + \|v_t^n\|_{r+1}^{r+1} + |\gamma u_t^n|_{q+1}^{q+1})d\tau \leq R, \end{cases}$$

where T can be arbitrarily large if $p \leq \min\{m, r\}$ and $k \leq q$, or T is sufficiently small if $p > \min\{m, r\}$ or $k > q$.

Now, put $y^n(t) = u(t) - u^n(t)$, $z^n(t) = v(t) - v^n(t)$, and

(6.3) $$\tilde{E}_n(t) = \frac{1}{2}(\|y^n(t)\|_{1,\Omega}^2 + \|z^n(t)\|_{1,\Omega}^2 + \|y_t^n(t)\|_2^2 + \|z_t^n(t)\|_2^2),$$

for $t \in [0, T]$. We aim to show $\tilde{E}_n(t) \longrightarrow 0$ uniformly on $[0, T]$, for sufficiently small T.

We begin by following the proof of Theorem 1.5, where here u, v, u^n, v^n, y^n, z^n, $\tilde{E}_n$ replace u, v, $\hat{u}$, $\hat{v}$, y, z, $\tilde{E}$ in the proof of Theorem 1.5, respectively. However,

due to having nonzero initial data, $y^n(0) = u_0 - u_0^n$ and $z^n(0) = v_0 - v_0^n$, we have to take care of the additional terms resulting from integration by parts.

First, as in (4.7), accounting for the nonzero initial data, we obtain the energy inequality

$$(6.4) \qquad \tilde{E}_n(t) \leq \tilde{E}_n(0) + R_f^n + R_h^n,$$

where

$$R_f^n = \int_0^t \int_\Omega (f_1(u,v) - f_1(u^n, v^n)) y_t^n dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(u^n, v^n)) z_t^n dx d\tau$$

and

$$R_h^n = \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma u^n)) \gamma y_t^n d\Gamma d\tau.$$

As in Step 2 in the proof of Theorem 1.5, the estimate for R_f^n when $1 \leq p \leq 3$ is straightforward. Indeed, following (4.9)-(4.10), we find

$$(6.5) \qquad R_f^n \leq C(R) \int_0^t \tilde{E}_n(\tau) d\tau.$$

If $3 < p < 5$, we utilize Assumption 1.4, and integration by parts in (4.12)-(4.13) yields the additional terms:

$$Q_1 = \left| \int_\Omega (f_1(u_0, v_0) - f_1(u_0^n, v_0^n)) y^n(0) dx \right| + \left| \int_\Omega (f_2(u_0, v_0) - f_2(u_0^n, v_0^n)) z^n(0) dx \right|,$$

which must be added to the right hand side of (4.13). Another place where we pick up additional nonzero terms is in (4.16); namely, the terms

$$Q_2 = \left| \int_\Omega \left(\frac{1}{2} \partial_u f_1(u_0, v_0) |y^n(0)|^2 + \partial_{uv}^2 F(u_0, v_0) y^n(0) z^n(0) \right. \right.$$
$$\left. \left. + \frac{1}{2} \partial_v f_2(u_0, v_0) |z^n(0)|^2 \right) dx \right|$$

must be added to the right hand side of (4.16).

By using (4.11), we deduce

$$Q_1 + Q_2$$
$$(6.6) \quad \leq C \int_\Omega (|u_0|^{p-1} + |u_0^n|^{p-1} + |v_0|^{p-1} + |v_0^n|^{p-1} + 1)(|y^n(0)|^2 + |z^n(0)|^2) dx.$$

A typical term on the right hand side of (6.6) is estimated in the following manner. By using Hölder's inequality and (6.2), we have

$$(6.7)$$
$$\int_\Omega |u_0^n|^{p-1} |y^n(0)|^2 dx \leq \|u_0^n\|_{\frac{3(p-1)}{2}}^{p-1} \|y^n(0)\|_6^2 \leq C(R) \|y^n(0)\|_{1,\Omega}^2 \leq C(R) \tilde{E}_n(0).$$

Thus,

$$(6.8) \qquad Q_1 + Q_2 \leq C(R) \tilde{E}_n(0).$$

The nonzero initial data, $y^n(0) \neq 0$ and $z^n(0) \neq 0$, also changes the estimates in (4.19)-(4.20). Indeed,

$$\int_\Omega |y^n(t)|^2 dx = \int_\Omega \left| y^n(0) + \int_0^t y_t^n(\tau)d\tau \right|^2 dx$$

$$\leq 2\int_\Omega |y^n(0)|^2 dx + 2\int_\Omega \left| \int_0^t y_t^n(\tau)d\tau \right|^2 dx$$

$$\leq C\left(\|y^n(0)\|_{1,\Omega}^2 + t\int_0^t \|y_t^n(\tau)\|_2^2\, d\tau \right)$$

$$(6.9) \qquad \leq C\left(\tilde{E}_n(0) + T\int_0^t \tilde{E}_n(\tau)d\tau \right).$$

Also, since the integral $\int_\Omega |z^n(t)|^2 dx$ can be estimated as in (6.9), we conclude that

$$(6.10) \qquad \int_\Omega \left(|y^n(t)|^2 + |z^n(t)|^2 \right) dx \leq C\left(\tilde{E}_n(0) + T\int_0^t \tilde{E}_n(\tau)d\tau \right).$$

Another place where one must exercise caution in estimating the typical term is in $\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx$. As in the proof of Theorem 1.5, we consider two cases, $3 < p < 5$ and $5 \leq p < 6$.

If $3 < p < 5$, then by using (4.25), (4.26) and (6.9), we obtain for $0 < \epsilon < 5 - p$:

$$(6.11) \quad \int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \leq 2\epsilon\tilde{E}(t) + C(\epsilon, R)\tilde{E}_n(0) + C(\epsilon, R, T)\int_0^t \tilde{E}(\tau)d\tau.$$

If $5 \leq p < 6$, the nonzero initial data make the computations more involved than (4.28)-(4.32). Recall the choice of $\phi \in C_0(\Omega)$ such that $\|u_0 - \phi\|_{\frac{3(p-1)}{2}} \leq \epsilon^{\frac{1}{p-1}}$, where the value of $\epsilon > 0$ will be chosen later. Then, we have

$$(6.12)$$
$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx \leq C\Big(\int_\Omega |u^n(t) - u_0^n|^{p-1}|y^n(t)|^2 dx$$
$$+ \int_\Omega |u_0^n - u_0|^{p-1}|y^n(t)|^2 dx + \int_\Omega |u_0 - \phi|^{p-1}|y^n(t)|^2 dx + \int_\Omega |\phi|^{p-1}|y^n(t)|^2 dx \Big).$$

As in (4.29), we deduce that

$$(6.13) \qquad \int_\Omega |u^n(t) - u_0^n|^{p-1}|y^n(t)|^2 dx \leq C(R)T^{\frac{m(p-1)}{m+1}}\tilde{E}_n(t).$$

Also, by using Hölder's inequality and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$(6.14) \qquad \int_\Omega |u_0^n - u_0|^{p-1}|y^n(t)|^2 dx \leq \|u_0^n - u_0\|_{\frac{3(p-1)}{2}}^{p-1} \|y^n(t)\|_6^2 \leq \epsilon\tilde{E}_n(t),$$

for all sufficiently large n, since $u_0^n \longrightarrow u_0$ in $L^{\frac{3(p-1)}{2}}(\Omega)$. Moreover, from (4.30), we know

$$(6.15) \qquad \int_\Omega |u_0 - \phi|^{p-1}|y^n(t)|^2 dx \leq C\epsilon\tilde{E}_n(t).$$

As for the last term on the right hand side of (6.12), we refer to (4.31) and (6.9), and we have

(6.16)
$$\int_\Omega |\phi|^{p-1}|y^n(t)|^2 dx \leq C(\epsilon)\int_\Omega |y^n(t)|^2 dx \leq C(\epsilon)\left(\tilde{E}_n(0) + T\int_0^t \tilde{E}_n(\tau)d\tau\right).$$

Thus, for the case $5 \leq p < 6$, it follows from (6.13)-(6.16) and (6.12) that

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx$$

(6.17)
$$\leq C(\epsilon)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t) + C(\epsilon)T\int_0^t \tilde{E}_n(\tau)d\tau.$$

By combining the two cases (6.11) and (6.17), we have for $3 < p < 6$:

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2 dx$$

(6.18)
$$\leq C(\epsilon, R)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{E}_n(t) + C(\epsilon, R, T)\int_0^t \tilde{E}_n(\tau)d\tau.$$

Now, by looking at (6.8), (6.10) and (6.18), we notice that the nonzero initial data $y^n(0)$ and $z^n(0)$ also contribute the additional term $C(\epsilon, R)\tilde{E}_n(0)$, which should be added to the right hand side of R_f, and so, for $3 < p < 6$ we have:

$$R_f^n \leq C(\epsilon, R)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\right)\tilde{E}_n(t)$$

(6.19)
$$+ C(\epsilon, R, T)\int_0^t \tilde{E}_n(\tau)\left(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1} + 1\right)d\tau,$$

for all sufficiently large n, and $\epsilon > 0$ sufficiently small, and according to (6.5), the estimate (6.19) also holds for the case $1 \leq p \leq 3$.

By using a similar approach (which is omitted) we can estimate R_h^n in (6.4) as well. Finally, from (6.4), we conclude that

$$\tilde{E}_n(t) \leq \tilde{E}_n(0) + R_f^n + R_h^n$$

$$\leq C(\epsilon, R)\tilde{E}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right)\tilde{E}_n(t)$$

$$+ C(\epsilon, R, T)\int_0^t \tilde{E}_n(\tau)\Big(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1}$$

$$+ |u_t|_{q+1} + |u_t^n|_{q+1} + 1\Big)d\tau.$$

Again, we can choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right) < 1.$$

By Gronwall's inequality, we obtain

$$\tilde{E}_n(t) \leq C(\epsilon, R, T)\tilde{E}_n(0)\exp\Big[\int_0^t \Big(\|u_t\|_{m+1} + \|v_t\|_{r+1}$$

(6.20)
$$+ \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1} + |u_t|_{q+1} + |u_t^n|_{q+1} + 1\Big)d\tau\Big],$$

and so, by (6.2), we have

$$\tilde{E}_n(t) \leq C(\epsilon, R, T)\tilde{E}_n(0), \tag{6.21}$$

for all sufficiently large n. Hence, $\tilde{E}_n(t) \longrightarrow 0$ uniformly on $[0, T]$. This concludes the proof of Theorem 1.10. $\qquad\square$

7. Appendix

Proposition 7.1. *Let X and Y be Banach spaces and assume $A_1 : \mathcal{D}(A_1) \subset X \longrightarrow X^*$, $A_2 : \mathcal{D}(A_2) \subset Y \longrightarrow Y^*$ are single-valued maximal monotone operators. Then, the operator $A : \mathcal{D}(A_1) \times \mathcal{D}(A_2) \subset X \times Y \longrightarrow X^* \times Y^*$ defined by*

$$A \left(\begin{array}{c} x \\ y \end{array} \right)^{tr} = \left(\begin{array}{c} A_1(x) \\ A_2(y) \end{array} \right)^{tr}$$

is also maximal monotone.

Proof. The fact that A is monotone is trivial. In order to show that A is maximal monotone, assume $(x_0, y_0) \in X \times Y$ and $(x_0^*, y_0^*) \in X^* \times Y^*$ such that

$$\langle x - x_0, A_1(x) - x_0^* \rangle + \langle y - y_0, A_2(y) - y_0^* \rangle \geq 0, \tag{7.1}$$

for all $(x, y) \in \mathcal{D}(A_1) \times \mathcal{D}(A_2)$.

If $x_0 \in \mathcal{D}(A_1)$, then by taking $x = x_0$ in (7.1) and using the maximal monotonicity of A_2, we obtain $y_0 \in \mathcal{D}(A_2)$ and $y_0^* = A_2(y_0)$, and then we can put $y = y_0$ in (7.1) and conclude from the maximal monotonicity of A_1 that $x_0^* = A_1(x_0)$. Similarly, if $y_0 \in \mathcal{D}(A_2)$, then it follows that $x_0 \in \mathcal{D}(A_1)$, $x_0^* = A_1(x_0)$ and $y_0^* = A_2(y_0)$.

Now, if $x_0 \notin \mathcal{D}(A_1)$ and $y_0 \notin \mathcal{D}(A_2)$, then since A_1 and A_2 are both maximal monotone, there exist $x_1 \in \mathcal{D}(A_1)$, $y_1 \in \mathcal{D}(A_2)$ such that $\langle x_1 - x_0, A_1(x_1) - x_0^* \rangle < 0$ and $\langle y_1 - y_0, A_2(y_1) - y_0^* \rangle < 0$. Therefore, $\langle x_1 - x_0, A_1(x_1) - x_0^* \rangle + \langle y_1 - y_0, A_2(y_1) - y_0^* \rangle < 0$, which contradicts (7.1).

Therefore, we must have $x_0 \in \mathcal{D}(A_1)$ and $y_0 \in \mathcal{D}(A_2)$, with $x_0^* = A_1(x_0)$ and $y_0^* = A_2(y_0)$. Thus, A is maximal monotone. $\qquad\square$

Lemma 7.2. *Let X be a Banach space and $1 \leq p < \infty$. Then, $C_0((0, T); X)$ is dense in $L^p(0, T; X)$, where $C_0((0, T); X)$ denotes the space of continuous functions $u : (0, T) \longrightarrow X$ with compact support in $(0, T)$.*

Remark 7.3. The result is well known if $X = \mathbb{R}^n$. Although for a general Banach space X such a result is expected, we couldn't find a reference for it in the literature. Thus, we provide a proof for it.

Proof. Let $u \in L^p(0, T; X)$, $\epsilon > 0$ be given. By the definition of $L^p(0, T; X)$, there exists a simple function ϕ with values in X such that

$$\int_0^T \|\phi(t) - u(t)\|_X^p \, dt < \epsilon^p. \tag{7.2}$$

Say $\phi(t) = \sum_{j=1}^n x_j \chi_{E_j}(t)$, where $x_j \in X$ are distinct, each $x_j \neq 0$, and $E_j \subset (0, T)$ are Lebesgue measurable such that $E_j \cap E_k = \emptyset$, for all $j \neq k$.

By a standard result in analysis, for each E_j, there exists a finite disjoint sequence of open segments $\{I_{j,k}\}_{k=1}^{m_j}$ such that

$$(7.3) \qquad m\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) < \left(\frac{\epsilon}{2n \|x_j\|_X}\right)^p \quad \text{for } j = 1, 2, ..., n,$$

where m denotes the Lebesgue measure, and $E \triangle F$ is the symmetric difference of the sets E and F. In particular, we have

$$m\left(\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) \cap [0,T]\right) < \left(\frac{\epsilon}{2n \|x_j\|_X}\right)^p \quad \text{for } j = 1, 2, ..., n.$$

Let us note that $\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) \cap [0,T] = E_j \triangle \left(\bigcup_{k=1}^{m_j} I_{j,k} \cap [0,T]\right)$. So, we may assume, without loss of generality, that each $I_{j,k} \subset [0,T]$.

Now, if E, $F \subset [0,T]$ are Lebesgue measurable, then

$$(7.4)$$
$$\int_0^T |\chi_E(t) - \chi_F(t)|^p dt$$
$$= \int_{E\setminus F} |\chi_E(t) - \chi_F(t)|^p dt + \int_{F\setminus E} |\chi_E(t) - \chi_F(t)|^p dt + \int_{E\cap F} |\chi_E(t) - \chi_F(t)|^p dt$$
$$= \int_{E\setminus F} \chi_E(t) dt + \int_{F\setminus E} \chi_F(t) dt = m(E \triangle F).$$

Therefore, by (7.4) and (7.3),

$$(7.5) \quad \|x_j\|_X^p \int_0^T |\chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t)|^p dt = \|x_j\|_X^p \, m\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) < \left(\frac{\epsilon}{2n}\right)^p.$$

Since $I_{j,k} \subset [0,T]$, we can select $\delta_{j,k}$ such that $0 < \delta_{j,k} < \frac{1}{4}(b_{j,k} - a_{j,k})$, where $I_{j,k} = (a_{j,k}, b_{j,k})$. Choose $\delta > 0$ such that

$$(7.6) \quad \delta < \min\left\{\delta_{j,k}, \ \frac{1}{8(2n)^{p-1} \sum_{j=1}^n (\|x_j\|_X^p \, m_j)} \epsilon^p : k = 1, \ldots, m_j; j = 1, \ldots, n\right\}.$$

Now we define the functions $g_{j,k} \in C_0((0,T); \mathbb{R})$ such that $g_{j,k}(t) = 1$ on $[a_{j,k} + 2\delta, b_{j,k} - 2\delta]$, $g_{j,k}(t)$ is linear on $[a_{j,k} + \delta, a_{j,k} + 2\delta] \cup [b_{j,k} - 2\delta, b_{j,k} - \delta]$, and $g_{j,k}(t) = 0$ outside $[a_{j,k} + \delta, b_{j,k} - \delta]$. Let us notice that

$$(7.7)$$
$$\int_0^T \left|\sum_{k=1}^{m_j}(\chi_{I_{j,k}}(t) - g_{j,k}(t))\right|^p dt \leq \int_0^T \left(\sum_{k=1}^{m_j}(\chi_{(a_{j,k}, a_{j,k}+2\delta)}(t) + \chi_{(b_{j,k}-2\delta, b_{j,k})}(t))\right)^p dt$$
$$= \int_0^T \sum_{k=1}^{m_j}\left(\chi_{(a_{j,k}, a_{j,k}+2\delta)}(t) + \chi_{(b_{j,k}-2\delta, b_{j,k})}(t)\right) dt = \sum_{k=1}^{m_j} 4\delta = 4m_j\delta.$$

Finally, we define $g(t) = \sum_{j=1}^n x_j \sum_{k=1}^{m_j} g_{j,k}(t)$. Clearly, $g \in C_0((0,T); X)$. Then, (7.2) yields

$$(7.8)$$
$$\|u - g\|_{L^p(0,T;X)} \leq \|u - \phi\|_{L^p(0,T;X)} + \|\phi - g\|_{L^p(0,T;X)} < \epsilon + \|\phi - g\|_{L^p(0,T;X)}.$$

For $t \in (0, T)$, we note that

$$
\|\phi(t) - g(t)\|_X = \left\| \sum_{j=1}^{n} \left(x_j \chi_{E_j}(t) - x_j \sum_{k=1}^{m_j} g_{j,k}(t) \right) \right\|_X
$$

$$
= \left\| \sum_{j=1}^{n} \left(x_j \chi_{E_j}(t) - x_j \sum_{j=1}^{m_j} \chi_{I_{j,k}}(t) + x_j \sum_{j=1}^{m_j} \chi_{I_{j,k}}(t) - x_j \sum_{k=1}^{m_j} g_{j,k}(t) \right) \right\|_X
$$

$$
\leq \sum_{j=1}^{n} \|x_j\|_X \left| \chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t) \right| + \sum_{j=1}^{n} \|x_j\|_X \sum_{k=1}^{m_j} \left| \chi_{I_{j,k}}(t) - g_{j,k}(t) \right|.
$$

So, by Jensen's inequality and (7.5)-(7.7), we have

$$
\int_0^T \|\phi(t) - g(t)\|_X^p \, dt \leq (2n)^{p-1} \sum_{j=1}^{n} \|x_j\|^p \int_0^T |\chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t)|^p dt
$$

$$
+ (2n)^{p-1} \sum_{j=1}^{n} \|x_j\|^p \int_0^T \left(\sum_{k=1}^{m_j} |\chi_{I_{j,k}}(t) - g_{j,k}(t)| \right)^p dt
$$

$$
(7.9) \qquad < (2n)^{p-1} \sum_{j=1}^{n} \left(\frac{\epsilon}{2n} \right)^p + (2n)^{p-1} \sum_{j=1}^{n} \|x_j\|_X^p \, 4m_j \delta < \frac{1}{2}\epsilon^p + \frac{1}{2}\epsilon^p = \epsilon^p.
$$

Combining (7.9) with (7.8) yields $\|u - g\|_{L^p(0,T;X)} < 2\epsilon$. $\square$

References

[1] Keith Agre and M. A. Rammaha, *Systems of nonlinear wave equations with damping and source terms*, Differential Integral Equations **19** (2006), no. 11, 1235–1270. MR2278006 (2007i:35165)

[2] Claudianor O. Alves, Marcelo M. Cavalcanti, Valeria N. Domingos Cavalcanti, Mohammad A. Rammaha, and Daniel Toundykov, *On existence, uniform decay rates and blow up for solutions of systems of nonlinear wave equations with damping and source terms*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 3, 583–608, DOI 10.3934/dcdss.2009.2.583. MR2525769 (2011a:35317)

[3] Đặng Đình Áng and A. Pham Ngoc Dinh, *Mixed problem for some semilinear wave equation with a nonhomogeneous condition*, Nonlinear Anal. **12** (1988), no. 6, 581–592, DOI 10.1016/0362-546X(88)90016-8. MR945666 (89h:35207)

[4] Viorel Barbu, Yanqiu Guo, Mohammad A. Rammaha, and Daniel Toundykov, *Convex integrals on Sobolev spaces*, J. Convex Anal. **19** (2012), no. 3, 837–852. MR3013761

[5] Viorel Barbu, *Analysis and control of nonlinear infinite-dimensional systems*, Mathematics in Science and Engineering, vol. 190, Academic Press Inc., Boston, MA, 1993. MR1195128 (93j:49002)

[6] Lorena Bociu, *Local and global wellposedness of weak solutions for the wave equation with nonlinear boundary and interior sources of supercritical exponents and damping*, Nonlinear Anal. **71** (2009), no. 12, e560–e575, DOI 10.1016/j.na.2008.11.062. MR2671860 (2011d:35320)

[7] Lorena Bociu and Irena Lasiecka, *Blow-up of weak solutions for the semilinear wave equations with nonlinear boundary and interior sources and damping*, Appl. Math. (Warsaw) **35** (2008), no. 3, 281–304, DOI 10.4064/am35-3-3. MR2453534 (2009h:35273)

[8] Lorena Bociu and Irena Lasiecka, *Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping*, Discrete Contin. Dyn. Syst. **22** (2008), no. 4, 835–860, DOI 10.3934/dcds.2008.22.835. MR2434972 (2010b:35305)

[9] Lorena Bociu and Irena Lasiecka, *Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping*, J. Differential Equations **249** (2010), no. 3, 654–683, DOI 10.1016/j.jde.2010.03.009. MR2646044 (2011g:35251)

[10] Haïm Brézis, *Intégrales convexes dans les espaces de Sobolev*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), 1972, pp. 9–23 (1973) (French, with English summary). MR0341077 (49 #5827)

[11] Marcelo M. Cavalcanti, Valéria N. Domingos Cavalcanti, and Irena Lasiecka, *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping—source interaction*, J. Differential Equations **236** (2007), no. 2, 407–459, DOI 10.1016/j.jde.2007.02.004. MR2322019 (2008c:35189)

[12] Igor Chueshov, Matthias Eller, and Irena Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Partial Differential Equations **27** (2002), no. 9-10, 1901–1951, DOI 10.1081/PDE-120016132. MR1941662 (2003m:35034)

[13] Igor Chueshov and Irena Lasiecka, *Long-time behavior of second order evolution equations with nonlinear damping*, Mem. Amer. Math. Soc. **195** (2008), no. 912, viii+183. MR2438025 (2009i:37200)

[14] Vladimir Georgiev and Grozdena Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations **109** (1994), no. 2, 295–308, DOI 10.1006/jdeq.1994.1051. MR1273304 (95b:35141)

[15] Yanqiu Guo and Mohammad A. Rammaha, *Blow-up of solutions to systems of nonlinear wave equations with supercritical sources*, Applicable Analysis (2013), Vol. 92, No. 6, 1101–1115.

[16] Yanqiu Guo and Mohammad A. Rammaha, *Global existence and decay of energy to systems of wave equations with damping and supercritical sources*, Z. Angew. Math. Phys. **64** (2013), 621–658. MR3068842

[17] Ryo Ikehata, *Some remarks on the wave equations with nonlinear damping and source terms*, Nonlinear Anal. **27** (1996), no. 10, 1165–1175, DOI 10.1016/0362-546X(95)00119-G. MR1407454 (97i:35117)

[18] Konrad Jörgens, *Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen*, Math. Z. **77** (1961), 295–308 (German). MR0130462 (24 #A323)

[19] Herbert Koch and Irena Lasiecka, *Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity-full von Karman systems*, Evolution equations, semigroups and functional analysis (Milano, 2000), Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhäuser, Basel, 2002, pp. 197–216. MR1944164 (2003j:35199)

[20] Hideo Kubo and Masahito Ohta, *Critical blowup for systems of semilinear wave equations in low space dimensions*, J. Math. Anal. Appl. **240** (1999), no. 2, 340–360, DOI 10.1006/jmaa.1999.6585. MR1731649 (2001f:35266)

[21] John E. Lagnese, *Boundary stabilization of thin plates*, SIAM Studies in Applied Mathematics, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. MR1061153 (91k:73001)

[22] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations*, Academic Press, New York, 1969, Mathematics in Science and Engineering, Vol. 55-I. MR0379933 (52:837)

[23] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equations **6** (1993), no. 3, 507–533. MR1202555 (94c:35129)

[24] I. Lasiecka and R. Triggiani, *Sharp regularity theory for second order hyperbolic equations of Neumann type. I. L_2 nonhomogeneous data*, Ann. Mat. Pura Appl. (4) **157** (1990), 285–367, DOI 10.1007/BF01765322. MR1108480 (92e:35102)

[25] Irena Lasiecka, *Mathematical control theory of coupled PDEs*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 75, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. MR1879543 (2003a:93002)

[26] Irena Lasiecka and Daniel Toundykov, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal. **64** (2006), no. 8, 1757–1797, DOI 10.1016/j.na.2005.07.024. MR2197360 (2006k:35189)

[27] Howard A. Levine and James Serrin, *Global nonexistence theorems for quasilinear evolution equations with dissipation*, Arch. Rational Mech. Anal. **137** (1997), no. 4, 341–361, DOI 10.1007/s002050050032. MR1463799 (99b:34110)

[28] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), no. 3-4, 273–303. MR0402291 (53 #6112)

[29] David R. Pitts and Mohammad A. Rammaha, *Global existence and non-existence theorems for nonlinear wave equations*, Indiana Univ. Math. J. **51** (2002), no. 6, 1479–1509, DOI 10.1512/iumj.2002.51.2215. MR1948457 (2003j:35219)

[30] Petronela Radu, *Weak solutions to the Cauchy problem of a semilinear wave equation with damping and source terms*, Adv. Differential Equations **10** (2005), no. 11, 1261–1300. MR2175336 (2007a:35110)

[31] Mohammad A. Rammaha, *The influence of damping and source terms on solutions of nonlinear wave equations*, Bol. Soc. Parana. Mat. (3) **25** (2007), no. 1-2, 77–90, DOI 10.5269/bspm.v25i1-2.7427. MR2379676 (2008k:35328)

[32] Mohammad A. Rammaha and Sawanya Sakuntasathien, *Critically and degenerately damped systems of nonlinear wave equations with source terms*, Appl. Anal. **89** (2010), no. 8, 1201–1227, DOI 10.1080/00036811.2010.483423. MR2681440 (2011e:35198)

[33] Mohammad A. Rammaha and Sawanya Sakuntasathien, *Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms*, Nonlinear Anal. **72** (2010), no. 5, 2658–2683, DOI 10.1016/j.na.2009.11.013. MR2577827 (2011a:35320)

[34] Mohammad A. Rammaha and Theresa A. Strei, *Global existence and nonexistence for nonlinear wave equations with damping and source terms*, Trans. Amer. Math. Soc. **354** (2002), no. 9, 3621–3637 (electronic), DOI 10.1090/S0002-9947-02-03034-9. MR1911514 (2003f:35214)

[35] Michael Reed, *Abstract non-linear wave equations*, Lecture Notes in Mathematics, Vol. 507, Springer-Verlag, Berlin, 1976. MR0605679 (58 #29290)

[36] Sawanya Sakuntasathien, *Global well-posedness for systems of nonlinear wave equations*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Nebraska - Lincoln. MR2711412

[37] R. Seeley, *Interpolation in L^p with boundary conditions*, Studia Math. **44** (1972), 47–60. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I. MR0315432 (47 #3981)

[38] Irving Segal, *Non-linear semi-groups*, Ann. of Math. (2) **78** (1963), 339–364. MR0152908 (27 #2879)

[39] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997. MR1422252 (98c:47076)

[40] Daniel Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 1, 185–206. MR1633000 (99e:35129)

[41] Roger Temam, *Navier-Stokes equations*, 3rd ed., Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam, 1984. Theory and numerical analysis; With an appendix by F. Thomasset. MR769654 (86m:76003)

[42] Daniel Toundykov, *Optimal decay rates for solutions of a nonlinear wave equation with localized nonlinear dissipation of unrestricted growth and critical exponent source terms under mixed boundary conditions*, Nonlinear Anal. **67** (2007), no. 2, 512–544, DOI 10.1016/j.na.2006.06.007. MR2317185 (2008f:35257)

[43] Enzo Vitillaro, *Some new results on global nonexistence and blow-up for evolution problems with positive initial energy*, Rend. Istit. Mat. Univ. Trieste **31** (2000), no. suppl. 2, 245–275. Workshop on Blow-up and Global Existence of Solutions for Parabolic and Hyperbolic Problems (Trieste, 1999). MR1800451 (2001j:35210)

[44] Enzo Vitillaro, *A potential well theory for the wave equation with nonlinear source and boundary damping terms*, Glasg. Math. J. **44** (2002), no. 3, 375–395, DOI 10.1017/S0017089502030045. MR1956547 (2003k:35169)

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