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Global well-posedness for nonlinear wave equations with supercritical source and damping terms

Yanqiu Guo

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ABSTRACT

We prove the global well-posedness of weak solutions for nonlinear wave equations with supercritical source and damping terms on a three-dimensional torus $\mathbb{T}^3$ of the prototype

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u, \quad (x,t) \in \mathbb{T}^3 \times \mathbb{R}^+;$$

$$u(0) = u_0 \in H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3), \quad u_t(0) = u_1 \in L^2(\mathbb{T}^3),$$

where $1 \leq p \leq \min\{\frac{2}{3}m + \frac{5}{3}, m\}$. Notably, p is allowed to be larger than 6.

1. Introduction

1.1. The model

Wave equations under the influence of nonlinear damping and source terms have attracted considerable attention. The canonical equation of this type reads

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u, \tag{1.1}$$

where $m, p \geq 1$. A major interest for this topic lies in understanding the "competition" between the frictional damping term $|u_t|^{m-1}u_t$ and the energy-amplifying source term $|u|^{p-1}u$.

The purpose of this paper is to provide a suitable assumption on p and m, such that model (1.1) is globally well-posed for weak solutions defined in a three-dimensional periodic physical domain, and the source term $|u|^{p-1}u$ is allowed to have a "fast" growth rate $p \geq 6$.

Let us review some important results in the literature which are concerned with equation (1.1). Georgiev and Todorova [12] studied (1.1) in a bounded domain $\Omega \subset \mathbb{R}^3$ with a Dirichlét boundary condition. For

E-mail address: yanguo@fiu.edu.

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a source term of subcritical or critical growth rate ($1 \le p \le 3$), they proved the well-posedness of weak solutions for (1.1). In addition, the solution is global if the damping dominates the source term in the sense that $m \ge p$, whereas the solution blows up in finite time if the strength of the source exceeds the damping effect, namely, $p > m$.

We remark that $p = 3$ is called the critical growth rate for the source term $|u|^{p-1}u$ because the operator $u \mapsto u^3$ is locally Lipschitz continuous from H^1 to L^2 in three dimensions.

Bociu and Lasiecka [6–8] considered (1.1) with supercritical source terms, in a bounded domain $\Omega \subset \mathbb{R}^3$ satisfying a Newman boundary condition, and showed the existence and uniqueness of weak solutions if $1 \le p \le \frac{6m}{m+1}$, allowing p have the range $[1, 6)$.

In the literature, the Cauchy problem for (1.1) in $\mathbb{R}^n$ was also investigated (see, e.g., [23,25]). Moreover, it is of interest to consider interaction between source terms and other types of damping terms in nonlinear wave equations, for instance, strong damping (e.g., [11]), degenerate damping (e.g., [4,5]), and viscoelastic damping (e.g. [17–19]). One may also refer to [1,3,9,14–16,22,24,26,27] and references therein for more works on nonlinear wave equations with damping and source terms. It is also worth mentioning papers [20,21] on analyticity for a class of nonlinear wave equations including (1.1) as a special case.

1.2. Main results

In this paper, we study the following nonlinear wave equation with damping and source terms defined in a three-dimensional fundamental periodic domain $\mathbb{T}^3 = [-\pi, \pi]^3$:

$$u_{tt} - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u, \quad (x,t) \in \mathbb{T}^3 \times \mathbb{R}^+; \tag{1.2}$$

$$u(x,0) = u_0(x) \in H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3), \quad u_t(x,0) = u_1(x) \in L^2(\mathbb{T}^3), \tag{1.3}$$

where $m, p \ge 1$. Our main result states, if $1 \le p \le \min\{\frac{2}{3}m + \frac{5}{3}, m\}$, then system (1.2)-(1.3) admits a unique global weak solution which depends continuously on initial data. Note, in the initial condition (1.3), we demand an extra integrability for u_0, namely, $u_0 \in L^{m+1}(\mathbb{T}^3)$ if $m > 5$.

We choose the physical domain to be a torus $\mathbb{T}^3$ because we want to focus on the interaction between the damping and source terms, without influence of boundary conditions. Also, we restrict our analysis to 3D since it is more physically relevant. Our results extend easily to an n-dimensional torus $\mathbb{T}^n$, by accounting for the corresponding Sobolev imbeddings, and accordingly adjusting the conditions imposed on the parameters.

Throughout the paper, we denote by $\|\cdot\|_s = \|\cdot\|_{L^s(\mathbb{T}^3)}$ for L^s-norm. Also, for a function $y(x,t)$ defined on $\mathbb{T}^3 \times \mathbb{R}^+$, the partial derivative in t is denoted by $y' = y_t = \frac{\partial y}{\partial t}$. Recall the gradient $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^{tr}$. Moreover, the space $H^1(\mathbb{T}^3)$ is defined as $H^1(\mathbb{T}^3) = \{f \in L^2(\mathbb{T}^3) : \nabla f \in L^2(\mathbb{T}^3)\}$ with its norm $\|f\|_{H^1(\mathbb{T}^3)} = (\|f\|_2^2 + \|\nabla f\|_2^2)^{1/2}$.

Let us introduce the definition of a weak solution for system (1.2)-(1.3).

Definition 1.1. Let $T > 0$. We call (u, u_t) a *weak solution* for system (1.2)-(1.3) on $[0, T]$ if

- $u(x,0) = u_0(x) \in H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)$, $u_t(x,0) = u_1(x) \in L^2(\mathbb{T}^3)$;
- $u \in L^\infty(0,T; H^1(\mathbb{T}^3)) \cap L^\infty(0,T; L^{m+1}(\mathbb{T}^3))$;
  $u_t \in L^\infty(0,T; L^2(\mathbb{T}^3)) \cap L^{m+1}(\mathbb{T}^3 \times (0,T))$;
  $u_{tt} \in L^{\frac{m+1}{m}}(0,T; X')$ where $X = H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)$;
- (u, u_t) verifies the identity

$$\int_{\mathbb{T}^3} u_t(t)\phi(t)dx - \int_{\mathbb{T}^3} u_t(0)\phi(0)dx - \int_0^t \int_{\mathbb{T}^3} u_t(\tau)\phi_t(\tau)dxd\tau + \int_0^t \int_{\mathbb{T}^3} \nabla u(\tau) \cdot \nabla\phi(\tau)dxd\tau$$

$$+\int_0^t\int_{\mathbb{T}^3}|u_t(\tau)|^{m-1}u_t(\tau)\phi(\tau)dxd\tau=\int_0^t\int_{\mathbb{T}^3}|u(\tau)|^{p-1}u(\tau)\phi(\tau)dxd\tau, \tag{1.4}$$

for all $t\in[0,T]$, and for any $\phi\in C([0,T];H^1(\mathbb{T}^3))\cap L^{m+1}(\mathbb{T}^3\times(0,T))$ with $\phi_t\in C([0,T];L^2(\mathbb{T}^3))$.

Our first theorem deals with the global existence of weak solution for the initial value problem (1.2)-(1.3). Also the energy identity holds for weak solutions. Moreover, a global solution (u,u_t) grows at most exponentially in time.

Theorem 1.2 *(Global existence of weak solutions). Assume either* Case 1*: $1\le p\le m\le 5$ or* Case 2*: $1\le p<\frac{5}{6}(m+1)$ for $m>5$. Suppose $u_0\in H^1(\mathbb{T}^3)\cap L^{m+1}(\mathbb{T}^3)$ and $u_1\in L^2(\mathbb{T}^3)$. Let $T>0$ be arbitrarily large. Then, system (1.2)-(1.3) has a weak solution (u,u_t) on $[0,T]$ in the sense of Definition 1.1. Also, the energy identity holds:*

$$E(t)+\int_0^t\|u_t(\tau)\|_{m+1}^{m+1}d\tau=E(0),\quad \textit{for all } t\in[0,T], \tag{1.5}$$

where the total energy $E(t):=\frac{1}{2}(\|\nabla u(t)\|_2^2+\|u_t(t)\|_2^2)-\frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}$. In addition,

$$\mathscr{E}(t)+\frac{1}{2}\int_0^t\|u_t(\tau)\|_{m+1}^{m+1}d\tau\le(\mathscr{E}(0)+t)\,e^{Ct},\quad \textit{for all } t\in[0,T], \tag{1.6}$$

where $\mathscr{E}(t):=\frac{1}{2}\left(\|\nabla u(t)\|_2^2+\|u_t(t)\|_2^2\right)+\frac{1}{m+1}\|u(t)\|_{m+1}^{m+1}$.

Our second theorem establishes the uniqueness of weak solutions by assuming a slightly stronger restriction on (m,p). Continuous dependence on initial data is also provided.

Theorem 1.3 *(Uniqueness and continuous dependence). Assume either* Case I*: $1\le p\le m\le 5$ or* Case II*: $1\le p\le\frac{2}{3}m+\frac{5}{3}$ for $m>5$. Suppose $u_0\in H^1(\mathbb{T}^3)\cap L^{m+1}(\mathbb{T}^3)$ and $u_1\in L^2(\mathbb{T}^3)$. Let $T>0$ be arbitrarily large. Then, system (1.2)-(1.3) has a unique weak solution (u,u_t) on $[0,T]$ in the sense of Definition 1.1. Also, the weak solution depends continuously on initial data. More precisely, let (u_0^n,u_1^n) be a sequence of initial data such that $\lim_{n\to\infty}\|u_0^n-u_0\|_{H^1}=0$, $\lim_{n\to\infty}\|u_0^n-u_0\|_{m+1}=0$ and $\lim_{n\to\infty}\|u_1^n-u_1\|_2=0$, then the corresponding sequence of weak solutions (u_n,u_n') converges to (u,u_t) in the sense that*

$$\lim_{n\to\infty}\left[\sup_{t\in[0,T]}\left(\|u_n-u\|_{H^1}^2+\|u_n-u\|_{m+1}^{m+1}+\|u_n'-u_t\|_2^2\right)\right]=0. \tag{1.7}$$

Remark 1.4. The range of (m,p) assumed in Theorem 1.3 (i.e. the "union" of Case I and Case II) can be equivalently expressed as $1\le p\le\min\{\frac{2}{3}m+\frac{5}{3},m\}$. Also, Case II of Theorem 1.3 is a slightly smaller region in the (m,p) plane compared to Case 2 of Theorem 1.2.

Remark 1.5. For the sake of clarity, we consider the "typical" frictional damping term $|u_t|^{m-1}u_t$ and the "typical" source term $|u|^{p-1}u$. Nonetheless, our results hold for more general damping and source terms. More precisely, the damping term $|u_t|^{m-1}u_t$ can be generalized to $g(u_t)$ where $g\in C(\mathbb{R})$ is a monotone increasing function vanishing at the origin such that

$$a|s|^{m+1}\le g(s)s\le b|s|^{m+1},\quad \text{where } b\ge a>0 \text{ and } m\ge 1.$$

Also, the source term $|u|^{p-1}u$ can be generalized to $h(u)$ where h is a $C^1(\mathbb{R})$ function (C^2 is required if $p > 3$) satisfying

$$\begin{cases} |h'(s)| \le C(|s|^{p-1}+1), & \text{if } 1 \le p \le 3; \\ |h''(s)| \le C(|s|^{p-2}+1), & \text{if } p > 3. \end{cases}$$

2. Global existence of weak solutions

This section is devoted to proving Theorem 1.2, namely, the existence of global weak solutions, the energy identity, and the exponential bound for the growth of solutions at large time.

2.1. Galerkin approximation system

We show the existence of weak solutions for system (1.2)-(1.3) via the standard Galerkin approximation method. Let us first review some classical results regarding Fourier series on a torus. For a periodic function $f \in L^1(\mathbb{T}^3)$ where $\mathbb{T}^3 = [-\pi, \pi]^3$, the kth Fourier coefficient of f is defined by $\hat{f}(k) = \int_{\mathbb{T}^3} f(x)e^{-ik\cdot x}dx$. The Fourier series of f at $x \in \mathbb{T}^3$ is written as $\sum_{k\in\mathbb{Z}^3} \hat{f}(k)e^{ik\cdot x}$. We define the square partial sum of the Fourier series of f by

$$P_n f(x) = \sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{Z}^3 \\ |k_1|,|k_2|,|k_3|\le n}} \hat{f}(k)e^{ik\cdot x}. \tag{2.1}$$

Note that, for a Fourier series, the square partial sum defined in (2.1) is different from the spherical partial sum: $\sum_{|k|\le n} \hat{f}(k)e^{ik\cdot x}$. It is a classical result that the square partial sum $P_n f$ converges to f in $L^s(\mathbb{T}^3)$ for any $s \in (1, \infty)$ (see, e.g., [13]), namely, for an $f \in L^s(\mathbb{T}^3)$ with $1 < s < \infty$,

$$\lim_{n\to\infty} \|P_n f - f\|_{L^s(\mathbb{T}^3)} = 0. \tag{2.2}$$

Moreover, for any $f \in L^s(\mathbb{T}^3)$ with $1 < s < \infty$,

$$\|P_n f\|_{L^s(\mathbb{T}^3)} \le c_s \|f\|_{L^s(\mathbb{T}^3)}, \tag{2.3}$$

for some positive constant c_s independent of n and f.

We consider the Galerkin approximation system

$$u_n'' - \Delta u_n + P_n(|u_n'|^{m-1}u_n') = P_n(|u_n|^{p-1}u_n), \quad (x,t) \in \mathbb{T}^3 \times \mathbb{R}^+; \tag{2.4}$$

$$u_n(x,0) = P_n u_0(x), \quad u_n'(x,0) = P_n u_1(x), \tag{2.5}$$

where $u_n(x,t)$ is a trigonometric polynomial of the form:

$$u_n(x,t) = \sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{Z}^3 \\ |k_1|,|k_2|,|k_3|\le n}} \hat{u}_n(k,t)e^{ik\cdot x}.$$

By the Cauchy-Peano theorem, for each n, Galerkin system (2.4)-(2.5) has a solution u_n on $[0, T_n)$ for some $T_n \in (0, \infty]$ which stands for the maximum life span.

2.2. Energy estimate

In this subsection, we show that (u_n, u_n') is bounded in the energy space $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ uniformly in n. Multiply (2.4) by u_n' and integrate over $\mathbb{T}^3 \times (0,t)$. One has, for $t \in [0, T_n)$,

$$\begin{aligned}&\frac{1}{2}\left(\|\nabla u_n(t)\|_2^2 + \|u_n'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau \\ &= \frac{1}{2}\left(\|\nabla u_n(0)\|_2^2 + \|u_n'(0)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} |u_n|^{p-1} u_n u_n' dx d\tau. \end{aligned} \tag{2.6}$$

Define a *modified* energy:

$$\mathscr{E}_n(t) = \frac{1}{2}\left(\|\nabla u_n(t)\|_2^2 + \|u_n'(t)\|_2^2\right) + \frac{1}{m+1}\|u_n(t)\|_{m+1}^{m+1} \geq 0. \tag{2.7}$$

Then (2.6) can be written as

$$\begin{aligned}&\mathscr{E}_n(t) + \int_0^t \int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau \\ &= \mathscr{E}_n(0) + \int_0^t \int_{\mathbb{T}^3} |u_n|^{p-1} u_n u_n' dx d\tau + \frac{1}{m+1}\int_{\mathbb{T}^3} \left(|u_n(t)|^{m+1} - |u_n(0)|^{m+1}\right) dx. \end{aligned} \tag{2.8}$$

Since, for $m \geq 1$,

$$\begin{aligned}&\frac{1}{m+1}\int_{\mathbb{T}^3} \left(|u_n(t)|^{m+1} - |u_n(0)|^{m+1}\right) dx \\ &= \int_{\mathbb{T}^3} \int_0^t \frac{d}{d\tau}\left(\frac{1}{m+1}|u_n(\tau)|^{m+1}\right) d\tau dx = \int_{\mathbb{T}^3} \int_0^t |u_n(\tau)|^{m-1} u_n(\tau) u_n'(\tau) d\tau dx,\end{aligned}$$

it follows from (2.8) that

$$\begin{aligned}&\mathscr{E}_n(t) + \int_0^t \int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau \\ &= \mathscr{E}_n(0) + \int_0^t \int_{\mathbb{T}^3} |u_n|^{p-1} u_n u_n' dx d\tau + \int_0^t \int_{\mathbb{T}^3} |u_n|^{m-1} u_n u_n' dx d\tau. \end{aligned} \tag{2.9}$$

We estimate the integrals on the right-hand side of (2.9). Since $m \geq p \geq 1$, by using Hölder's inequality and Young's inequality, we obtain

$$\int_0^t \int_{\mathbb{T}^3} |u_n|^p |u_n'| dx d\tau \leq C t^{\frac{m-p}{m+1}} \left(\int_0^t \int_{\mathbb{T}^3} |u_n|^{m+1} dx d\tau\right)^{\frac{p}{m+1}} \left(\int_0^t \int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau\right)^{\frac{1}{m+1}}$$

$$\leq \frac{1}{4}\int_0^t\int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau + C\int_0^t\int_{\mathbb{T}^3} |u_n|^{m+1} dx d\tau + t. \tag{2.10}$$

Similarly, we have

$$\int_0^t\int_{\mathbb{T}^3} |u_n|^m |u_n'| dx d\tau \leq \frac{1}{4}\int_0^t\int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau + C\int_0^t\int_{\mathbb{T}^3} |u_n|^{m+1} dx d\tau. \tag{2.11}$$

Substituting (2.10)-(2.11) into (2.9) yields

$$\begin{aligned}\mathscr{E}_n(t) + \frac{1}{2}\int_0^t\int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau &\leq \mathscr{E}_n(0) + C\int_0^t\int_{\mathbb{T}^3} |u_n|^{m+1} dx d\tau + t \\ &\leq \mathscr{E}_n(0) + C\int_0^t \mathscr{E}_n(\tau) d\tau + t, \quad \text{for all } t \in [0, T_n).\end{aligned}$$

Then, using Grönwall's inequality, we obtain

$$\mathscr{E}_n(t) + \frac{1}{2}\int_0^t\int_{\mathbb{T}^3} |u_n'|^{m+1} dx d\tau \leq (\mathscr{E}_n(0) + t)\, e^{Ct}, \quad \text{for all } \ t \in [0, T_n). \tag{2.12}$$

By (2.7),

$$\mathscr{E}_n(0) = \frac{1}{2}(\|\nabla(P_n u_0)\|_2^2 + \|P_n u_1\|_2^2) + \frac{1}{m+1}\|P_n u_0\|_{m+1}^{m+1}.$$

According to Plancherel's theorem, we have $\|P_n u_1\|_2^2 \leq \|u_1\|_2^2$ for all $n \in \mathbb{N}$ and $\|\nabla(P_n u_0)\|_2^2 = \|P_n(\nabla u_0)\|_2^2 \leq \|\nabla u_0\|_2^2$ for all $n \in \mathbb{N}$. Moreover, since $u_0 \in L^{m+1}(\mathbb{T}^3)$ with $m \geq 1$, by virtue of (2.3), we have $\|P_n u_0\|_{m+1} \leq c_m \|u_0\|_{m+1}$, for some constant c_m independent of n and u_0. Therefore, $\mathscr{E}_n(0)$ has an upper bound independent of n, namely

$$\mathscr{E}_n(0) \leq \frac{1}{2}(\|\nabla u_0\|_2^2 + \|u_1\|_2^2) + c_m \|u_0\|_{m+1}^{m+1}. \tag{2.13}$$

Applying (2.13) to the right-hand side of (2.12) yields, for any $n \in \mathbb{N}$,

$$\begin{aligned}&\frac{1}{2}\left(\|\nabla u_n(t)\|_2^2 + \|u_n'(t)\|_2^2\right) + \frac{1}{m+1}\|u_n(t)\|_{m+1}^{m+1} + \frac{1}{2}\int_0^t \|u_n'(\tau)\|_{m+1}^{m+1} d\tau \\ &\leq \left(\frac{1}{2}(\|\nabla u_0\|_2^2 + \|u_1\|_2^2) + c_m\|u_0\|_{m+1}^{m+1} + t\right) e^{Ct},\end{aligned} \tag{2.14}$$

for all $t \in [0, T_n)$. Since $u_0 \in H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)$ and $u_1 \in L^2(\mathbb{T}^3)$, we see from (2.14) that a solution (u_n, u_n') for the Galerkin system (2.4)-(2.5) does not blow up at T_n. Therefore, we conclude $T_n = \infty$, namely, (u_n, u_n') is a global solution for (2.4) for all $t \in [0, \infty)$.

For $m \geq 1$, we have

$$\|u_n\|_{H^1(\mathbb{T}^3)}^2 = \|\nabla u_n\|_2^2 + \|u_n\|_2^2 \leq \|\nabla u_n\|_2^2 + C\|u_n\|_{m+1}^{m+1} + 1. \tag{2.15}$$

Let $T > 0$. Since (2.14) holds for all $t \geq 0$, then by (2.15), one has

$$u_n \text{ is uniformly bounded in } L^\infty(0,T; H^1(\mathbb{T}^3)). \tag{2.16}$$

Moreover, by (2.14), we have

$$u_n \text{ is uniformly bounded in } L^\infty(0,T; L^{m+1}(\mathbb{T}^3)); \tag{2.17}$$

$$u_n' \text{ is uniformly bounded in } L^\infty(0,T; L^2(\mathbb{T}^3)); \tag{2.18}$$

$$u_n' \text{ is uniformly bounded in } L^{m+1}(\mathbb{T}^3 \times (0,T)). \tag{2.19}$$

Notice $\int_{\mathbb{T}^3} \left||u_n|^{p-1}u_n\right|^{\frac{m+1}{m}} dx = \int_{\mathbb{T}^3} |u_n|^{\frac{(m+1)p}{m}} dx \leq C \left(\int_{\mathbb{T}^3} |u_n|^{m+1} dx\right)^{\frac{p}{m}}$ for $m \geq p \geq 1$. Thus, because of (2.17), we obtain

$$|u_n|^{p-1}u_n \text{ is uniformly bounded in } L^\infty(0,T; L^{\frac{m+1}{m}}(\mathbb{T}^3)). \tag{2.20}$$

Also, since $\int_0^T \int_{\mathbb{T}^3} \left||u_n'|^{m-1}u_n'\right|^{\frac{m+1}{m}} dxdt = \int_0^T \int_{\mathbb{T}^3} |u_n'|^{m+1} dxdt$, and due to (2.19), it follows that

$$|u_n'|^{m-1}u_n' \text{ is uniformly bounded in } L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T)). \tag{2.21}$$

Because of the Galerkin equation (2.4), we have $u_n'' = \Delta u_n - P_n(|u_n'|^{m-1}u_n') + P_n(|u_n|^{p-1}u_n)$. Note that Δu_n is uniformly bounded in $L^\infty(0,T; (H^1(\mathbb{T}^3))')$ due to (2.16). Thus, by virtue of (2.3), (2.20) and (2.21), we obtain

$$u_n'' \text{ is uniformly bounded in } L^{\frac{m+1}{m}}(0,T; X'), \text{ where } X = H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3). \tag{2.22}$$

By virtue of the uniform bounds (2.16)-(2.19) and (2.22), there exists a subsequence of u_n satisfying

$$u_n \to u \text{ weakly}^* \text{ in } L^\infty(0,T; H^1(\mathbb{T}^3)); \tag{2.23}$$

$$u_n \to u \text{ weakly}^* \text{ in } L^\infty(0,T; L^{m+1}(\mathbb{T}^3)); \tag{2.24}$$

$$u_n' \to u' \text{ weakly}^* \text{ in } L^\infty(0,T; L^2(\mathbb{T}^3)); \tag{2.25}$$

$$u_n' \to u' \text{ weakly in } L^{m+1}(\mathbb{T}^3 \times (0,T)); \tag{2.26}$$

$$u_n'' \to u'' \text{ weakly}^* \text{ in } L^{\frac{m+1}{m}}(0,T; X'), \tag{2.27}$$

where $X = H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)$.

Moreover, because of (2.16) and (2.18), and by the compact imbedding $H^1 \hookrightarrow H^{1-\epsilon} \hookrightarrow L^2$ for an $\epsilon \in (0,1)$, we conclude from the Aubin-Lions-Simon lemma that, on a subsequence

$$u_n \to u \text{ in } C([0,T]; H^{1-\epsilon}(\mathbb{T}^3)). \tag{2.28}$$

Furthermore, due to (2.28), one can extract a subsequence

$$u_n \to u \text{ almost everywhere in } \mathbb{T}^3 \times (0,T). \tag{2.29}$$

2.3. Convergence of the source term $|u_n|^{p-1}u_n$

We show $|u_n|^{p-1}u_n$ converges weakly to $|u|^{p-1}u$ in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$.

In fact, $\big||u_n|^{p-1}u_n - |u|^{p-1}u\big| \leq C(|u_n|^{p-1} + |u|^{p-1})|u_n - u|$, for $p \geq 1$. Thus, due to (2.29), we have

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \text{ almost everywhere in } \mathbb{T}^3 \times (0,T). \tag{2.30}$$

Recall a real analysis result: for a sequence of functions f_n defined on a measure space Y, if $\sup_n \|f_n\|_{L^s(Y)} < \infty$ and $f_n \to f$ a.e. in Y, then $f_n \to f$ weakly in $L^s(Y)$ if $1 < s < \infty$ (see, e.g., [10]). Here, by (2.20), we know $|u_n|^{p-1}u_n$ is uniformly bounded in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, thus along with (2.30), we conclude

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \text{ weakly in } L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T)). \tag{2.31}$$

2.4. Convergence of the damping term $|u_n'|^{m-1}u_n'$

In this section, we show that $|u_n'|^{m-1}u_n'$ converges weakly to $|u'|^{m-1}u'$ in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$. The monotonicity of the damping term is critical to our argument.

Thanks to (2.21), there exists a function $\psi \in L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$ and a subsequence of $|u_n'|^{m-1}u_n'$ such that

$$|u_n'|^{m-1}u_n' \longrightarrow \psi \text{ weakly in } L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T)). \tag{2.32}$$

It remains to show $\psi = |u'|^{m-1}u'$.

Set $w = u_n - u_j$. Due to the Galerkin system (2.4), the following equality is valid.

$$\begin{aligned}
&\frac{1}{2}\left(\|\nabla w(t)\|_2^2 + \|w'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left[P_n(|u_n'|^{m-1}u_n') - P_j(|u_j'|^{m-1}u_j')\right] w' dx d\tau \\
&= \frac{1}{2}\left(\|\nabla w(0)\|_2^2 + \|w'(0)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left[P_n(|u_n|^{p-1}u_n) - P_j(|u_j|^{p-1}u_j)\right] w' dx d\tau.
\end{aligned} \tag{2.33}$$

We remark that the projection P_n in the Galerkin system affects the monotonicity of the nonlinear damping term. Especially, $\int_{\mathbb{T}^3} \left[P_n(|u_n'|^{m-1}u_n') - P_j(|u_j'|^{m-1}u_j')\right] w' dx$ is not necessarily positive. To remedy the situation, we split this integral into a positive part and a "residue" part. More precisely, by assuming $n \geq j$, we have

$$\begin{aligned}
&\int_{\mathbb{T}^3} \left[P_n(|u_n'|^{m-1}u_n') - P_j(|u_j'|^{m-1}u_j')\right] w' dx = \int_{\mathbb{T}^3} \left[|u_n'|^{m-1}u_n' - P_j(|u_j'|^{m-1}u_j')\right] w' dx \\
&= \int_{\mathbb{T}^3} \left(|u_n'|^{m-1}u_n' - |u_j'|^{m-1}u_j'\right) w' dx + \int_{\mathbb{T}^3} \left[(P_n - P_j)(|u_j'|^{m-1}u_j')\right] w' dx.
\end{aligned} \tag{2.34}$$

In addition, for the sake of convenience, we also split the integral of source terms in (2.33) in the same manner as (2.34). As a result, if $n \geq j$, then equality (2.33) can be written as

$$\frac{1}{2}\left(\|\nabla w(t)\|_2^2 + \|w'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left(|u_n'|^{m-1}u_n' - |u_j'|^{m-1}u_j'\right) w' dx d\tau$$

$$+\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j'|^{m-1}u_j')\right]w'dxd\tau$$

$$=\frac{1}{2}\left(\|\nabla w(0)\|_2^2+\|w'(0)\|_2^2\right)+\int_0^t\int_{\mathbb{T}^3}\left(|u_n|^{p-1}u_n-|u_j|^{p-1}u_j\right)w'dxd\tau$$

$$+\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j|^{p-1}u_j)\right]w'dxd\tau. \tag{2.35}$$

Note $(|u_n'|^{m-1}u_n'-|u_j'|^{m-1}u_j')w'=(|u_n'|^{m-1}u_n'-|u_j'|^{m-1}u_j')(u_n'-u_j')\geq 0$ due to monotonicity of the function $|s|^{m-1}s$ for $m\geq 1$. Then we obtain from (2.35) that, for $n\geq j$,

$$0\leq\frac{1}{2}\left(\|\nabla w(t)\|_2^2+\|w'(t)\|_2^2\right)+\int_0^t\int_{\mathbb{T}^3}\left(|u_n'|^{m-1}u_n'-|u_j'|^{m-1}u_j'\right)w'dxd\tau$$

$$\leq\frac{1}{2}\left(\|\nabla w(0)\|_2^2+\|w'(0)\|_2^2\right)+\left|\int_0^t\int_{\mathbb{T}^3}\left(|u_n|^{p-1}u_n-|u_j|^{p-1}u_j\right)w'dxd\tau\right|$$

$$+\left|\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j|^{p-1}u_j)\right]w'dxd\tau\right|+\left|\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j'|^{m-1}u_j')\right]w'dxd\tau\right|. \tag{2.36}$$

2.4.1. Estimate for the "residue" terms

We estimate the two "residue" terms in (2.36). They are $\int_0^t\int_{\mathbb{T}^3}[(P_n-P_j)(|u_j'|^{m-1}u_j')]w'dxd\tau$ and $\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j|^{p-1}u_j)\right]w'dxd\tau$, for $n\geq j$. We aim to show that they approach zero when n and j are large. The estimates for these two integrals are essentially the same. Thus we present the estimate for $\int_0^t\int_{\mathbb{T}^3}[(P_n-P_j)(|u_j'|^{m-1}u_j')]w'dxd\tau$ in details only.

By the definition of P_n in (2.1), we know that for any functions f, $g\in L^2(\mathbb{T}^3)$ and any $n\in\mathbb{N}$,

$$\int_{\mathbb{T}^3}(P_nf)gdx=\sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{Z}^3\\|k_1|,|k_2|,|k_3|\leq n}}\hat{f}(k)\overline{\hat{g}(k)}=\int_{\mathbb{T}^3}f(P_ng)dx. \tag{2.37}$$

Formula (2.37) will be repeatedly used in the following calculations.

Since we assume $n\geq j$ in (2.35), then $P_nu_j'=u_j'=P_ju_j'$, i.e., $(P_n-P_j)u_j'=0$. Also, recall $w=u_n-u_j$. As a result,

$$\int_0^t\int_{\mathbb{T}^3}\left[(P_n-P_j)(|u_j'|^{m-1}u_j')\right]w'dxd\tau=\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')\left[(P_n-P_j)(u_n'-u_j')\right]dxd\tau$$

$$=\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')\left[(P_n-P_j)u_n'\right]dxd\tau$$

$$=\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')u_n'dxd\tau-\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')P_ju_n'dxd\tau. \tag{2.38}$$

Recall that $|u_j'|^{m-1}u_j' \to \psi$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3\times(0,T))$ by (2.32), and $u_n' \to u'$ weakly in $L^{m+1}(\mathbb{T}^3\times(0,T))$ by (2.26). Hence, for $0 \le t \le T$,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')u_n'dxd\tau = \lim_{j\to\infty}\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')u'dxd\tau$$
$$= \int_0^t\int_{\mathbb{T}^3}\psi u'dxd\tau. \tag{2.39}$$

Next, we look at the second term on the right-hand side of (2.38). Owing to (2.3), we have $\int_0^T\int_{\mathbb{T}^3}|P_j(|u_j'|^{m-1}u_j')|^{\frac{m+1}{m}}dxdt \le c_m\int_0^T\int_{\mathbb{T}^3}\left||u_j'|^{m-1}u_j'\right|^{\frac{m+1}{m}}dxdt = c_m\int_0^T\int_{\mathbb{T}^3}|u_j'|^{m+1}dxdt < \infty$ due to (2.19). Hence $P_j(|u_j'|^{m-1}u_j') \in L^{\frac{m+1}{m}}(\mathbb{T}^3\times(0,T))$. Consequently, since $u_n' \to u'$ weakly in $L^{m+1}(\mathbb{T}^3\times(0,T))$, then, for each fixed j, using (2.37),

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')P_ju_n'dxd\tau = \lim_{n\to\infty}\int_0^t\int_{\mathbb{T}^3}\left[P_j(|u_j'|^{m-1}u_j')\right]u_n'dxd\tau$$
$$= \int_0^t\int_{\mathbb{T}^3}\left[P_j(|u_j'|^{m-1}u_j')\right]u'dxd\tau, \tag{2.40}$$

for $0 \le t \le T$. Again, use (2.37) to get

$$\int_0^t\int_{\mathbb{T}^3}\left[P_j(|u_j'|^{m-1}u_j')\right]u'dxd\tau = \int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')P_ju'dxd\tau$$
$$= \int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')(P_ju'-u')dxd\tau + \int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')u'dxd\tau. \tag{2.41}$$

Since $u' \in L^{m+1}(\mathbb{T}^3\times(0,T))$ and $|u_j'|^{m-1}u_j' \to \psi$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3\times(0,T))$, one has

$$\lim_{j\to\infty}\int_0^t\int_{\mathbb{T}^3}(|u_j'|^{m-1}u_j')u'dxd\tau = \int_0^t\int_{\mathbb{T}^3}\psi u'dxd\tau, \tag{2.42}$$

for $0 \le t \le T$. Now we deal with the first integral on the right-hand side of (2.41). Since $u' \in L^{m+1}(\mathbb{T}^3\times(0,T))$, $\lim_{j\to\infty}\|P_ju'-u'\|_{m+1} = 0$ for a.e. $t \in [0,T]$, due to (2.2). Also, $\|P_ju'-u'\|_{m+1} \le \|P_ju'\|_{m+1} + \|u'\|_{m+1} \le c_m\|u'\|_{m+1}$ by (2.3). Hence, applying the Lebesgue's dominated convergence theorem, one has $\lim_{j\to\infty}\int_0^T\|P_ju'-u'\|_{m+1}^{m+1}dt = 0$. Then, employing the Hölder's inequality,

$$\int_0^T\int_{\mathbb{T}^3}|u_j'|^m|P_ju'-u'|dxdt \le \|u_j'\|_{L^{m+1}(\mathbb{T}^3\times(0,T))}\left(\int_0^T\|P_ju'-u'\|_{m+1}^{m+1}dt\right)^{\frac{1}{m+1}}$$
$$\le C\left(\int_0^T\|P_ju'-u'\|_{m+1}^{m+1}dt\right)^{\frac{1}{m+1}} \longrightarrow 0, \quad \text{as } j\to\infty, \tag{2.43}$$

where we have used the fact that u_j' is uniformly bounded in $L^{m+1}(\mathbb{T}^3 \times (0,T))$ by (2.19).

Combining (2.41), (2.42) and (2.43) gives

$$\lim_{j\to\infty} \int_0^t \int_{\mathbb{T}^3} \left[P_j(|u_j'|^{m-1}u_j')\right] u' dx d\tau = \int_0^t \int_{\mathbb{T}^3} \psi u' dx d\tau. \tag{2.44}$$

Then, by (2.40) and (2.44), one has

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_0^t \int_{\mathbb{T}^3} (|u_j'|^{m-1}u_j') P_j u_n' dx d\tau = \int_0^t \int_{\mathbb{T}^3} \psi u' dx d\tau, \quad \text{for all } t \in [0,T]. \tag{2.45}$$

Finally, by (2.38), (2.39) and (2.45), we conclude

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_0^t \int_{\mathbb{T}^3} \left[(P_n - P_j)(|u_j'|^{m-1}u_j')\right] w' dx d\tau = 0, \quad \text{for all } t \in [0,T]. \tag{2.46}$$

In the same manner, we can show

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_0^t \int_{\mathbb{T}^3} [(P_n - P_j)(|u_j|^{p-1}u_j)] w' dx d\tau = 0, \quad \text{for all } t \in [0,T]. \tag{2.47}$$

2.4.2. Estimate for the integral $\int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n - |u_j|^{p-1}u_j) w' dx d\tau$

Case A: $1 \le p < \frac{5}{6}(m+1)$.

We estimate

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^3} \left||u_n|^{p-1}u_n - |u_j|^{p-1}u_j\right| |w'| dx dt \\
&\le C \int_0^T \int_{\mathbb{T}^3} |u_n - u_j|(|u_n|^{p-1} + |u_j|^{p-1})|w'| dx dt \\
&\le C\|u_n - u_j\|_{L^{\frac{m+1}{m+1-p}}(\mathbb{T}^3\times(0,T))} \left(\|u_n\|^{p-1}_{L^{m+1}(\mathbb{T}^3\times(0,T))} + \|u_j\|^{p-1}_{L^{m+1}(\mathbb{T}^3\times(0,T))}\right) \|w'\|_{L^{m+1}(\mathbb{T}^3\times(0,T))},
\end{aligned} \tag{2.48}$$

where we use the Hölder's inequality in the last step.

Notice that $p < \frac{5}{6}(m+1)$ implies $\frac{m+1}{m+1-p} < 6$. So there exists $\epsilon_0 \in (0,1)$ such that $H^{1-\epsilon_0}(\mathbb{T}^3) \hookrightarrow L^{\frac{m+1}{m+1-p}}(\mathbb{T}^3)$. Then, by (2.28), there is a subsequence $u_n \to u$ in $C([0,T]; H^{1-\epsilon_0}(\mathbb{T}^3))$, which implies that $u_n \to u$ in $L^{\frac{m+1}{m+1-p}}(\mathbb{T}^3 \times (0,T))$. It follows that

$$\lim_{n,j\to\infty} \|u_n - u_j\|_{L^{\frac{m+1}{m+1-p}}(\mathbb{T}^3\times(0,T))} = 0. \tag{2.49}$$

Since $w' = u_n' - u_j'$ and u_n' is uniformly bounded in $L^{m+1}(\mathbb{T}^3 \times (0,T))$, one has w' is uniformly bounded in $L^{m+1}(\mathbb{T}^3 \times (0,T))$. Also, recall u_n is uniformly bounded in $L^\infty(0,T; L^{m+1}(\mathbb{T}^3))$ by (2.17). Then, by (2.48) and (2.49), we have

$$\lim_{n,j\to\infty}\int_0^T\int_{\mathbb{T}^3}\left||u_n|^{p-1}u_n-|u_j|^{p-1}u_j\right||w'|dxdt=0. \tag{2.50}$$

Applying (2.46), (2.47) and (2.50) to inequality (2.36), and since $\lim_{n,j\to\infty}(\|\nabla w(0)\|_2^2+\|w'(0)\|_2^2) = \lim_{n,j\to\infty}(\|\nabla(P_n-P_j)u_0\|_2^2+\|(P_n-P_j)u_1\|_2^2)=0$, we obtain

$$\lim_{j\to\infty}\overline{\lim_{n\to\infty}}\int_0^T\int_{\mathbb{T}^3}\left(|u_n'|^{m-1}u_n'-|u_j'|^{m-1}u_j\right)w'dxd\tau=0. \tag{2.51}$$

Remark 2.1. Recall Case 1 of the assumption for the "existence" result (Theorem 1.2) is that $1\le p\le m\le 5$. Notice

$$\{(m,p)\neq(5,5):\ 1\le p\le m\le 5\}\subset\{(m,p):1\le p<\frac{5}{6}(m+1)\}. \tag{2.52}$$

Nonetheless, the situation $m=p=5$ does not satisfy $1\le p<\frac{5}{6}(m+1)$, thus we have to discuss it separately.

Case B: $m=p=5$.

In this case,

$$\int_0^t\int_{\mathbb{T}^3}\left(|u_n|^{p-1}u_n-|u_j|^{p-1}u_j\right)w'dxd\tau=\int_0^t\int_{\mathbb{T}^3}\left(u_n^5-u_j^5\right)w'dxd\tau.$$

We estimate the above integral by using integration by parts in time. Such "integration by parts" technique originates from [6]. Indeed, since $w=u_n-u_j$,

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{T}^3}\left(u_n^5-u_j^5\right)w'dxd\tau\right|\\
&=\frac{1}{2}\left|\int_0^t\int_{\mathbb{T}^3}(u_n^4+u_n^3u_j+u_n^2u_j^2+u_nu_j^3+u_j^4)(w^2)'dxd\tau\right|\\
&\le C\int_{\mathbb{T}^3}\left(|u_n(t)|^4+|u_j(t)|^4\right)|w(t)|^2dx+C\int_{\mathbb{T}^3}\left(|u_n(0)|^4+|u_j(0)|^4\right)|w(0)|^2dx\\
&\quad+C\int_0^t\int_{\mathbb{T}^3}\left(|u_n|^3+|u_j|^3\right)\left(|u_n'|+|u_j'|\right)w^2dxd\tau.
\end{aligned} \tag{2.53}$$

We shall estimate each term on the right-hand side of (2.53).

Owing to (2.14) and (2.15), when $m=5$, there exists a uniform bound K such that

$$\|u_n(t)\|_{H^1}^2+\|u_n'(t)\|_2^2+\int_0^t\|u_n'(\tau)\|_6^6d\tau\le K,\ \ \text{for all } t\in[0,T],\ \text{for all } n\in\mathbb{N}. \tag{2.54}$$

Also, if one restricts $T\le 1$, then the bound K depends only on $\mathscr{E}(0)=\frac{1}{2}(\|\nabla u_0\|_2^2+\|u_1\|_2^2)+\frac{1}{6}\|u_0\|_6^6$.

By Hölder's inequality,

$$\int_0^t \int_{\mathbb{T}^3} \left(|u_n|^3 + |u_j|^3\right)\left(|u_n'| + |u_j'|\right) w^2 dx d\tau \le C \int_0^t \left(\|u_n\|_6^3 + \|u_j\|_6^3\right)\left(\|u_n'\|_6 + \|u_j'\|_6\right)\|w\|_6^2 d\tau$$

$$\le C(K) \int_0^t \left(\|u_n'\|_6 + \|u_j'\|_6\right)\|w\|_{H^1}^2 d\tau, \quad \text{for all } t \in [0,T], \tag{2.55}$$

where we use $H^1 \hookrightarrow L^6$ as well as (2.54) to obtain the last inequality.

Also, using Hölder's inequality, we have

$$\int_{\mathbb{T}^3} \left(|u_n(0)|^4 + |u_j(0)|^4\right)|w(0)|^2 dx$$
$$\le \left(\|u_n(0)\|_6^4 + \|u_j(0)\|_6^4\right)\|w(0)\|_6^2 \le C(K)\|w(0)\|_{H^1}^2, \tag{2.56}$$

where the last inequality is due to (2.54).

Next, we estimate $\int_{\mathbb{T}^3}\left(|u_n(t)|^4 + |u_j(t)|^4\right)|w(t)|^2 dx$. Indeed,

$$\int_{\mathbb{T}^3} |u_n(t)|^4 |w(t)|^2 dx \le C \int_{\mathbb{T}^3} |u_n(t) - u_n(0)|^4 |w(t)|^2 dx + C \int_{\mathbb{T}^3} |u_n(0) - u_0|^4 |w(t)|^2 dx$$
$$+ C \int_{\mathbb{T}^3} |u_0|^4 |w(t)|^2 dx. \tag{2.57}$$

We estimate each term on the right-hand side of (2.57). First,

$$\int_{\mathbb{T}^3} |u_n(t) - u_n(0)|^4 |w(t)|^2 dx = \int_{\mathbb{T}^3} \left|\int_0^t u_n'(\tau) d\tau\right|^4 |w(t)|^2 dx$$
$$\le \left(\int_{\mathbb{T}^3} \left|\int_0^t u_n'(\tau) d\tau\right|^6 dx\right)^{\frac{2}{3}} \|w(t)\|_6^2$$
$$\le C t^{\frac{10}{3}} \left(\int_0^t \|u_n'(\tau)\|_6^6 d\tau\right)^{\frac{2}{3}} \|w(t)\|_{H^1}^2 \le C(K) t^{\frac{10}{3}} \|w(t)\|_{H^1}^2, \quad \text{for all } t \in [0,T], \tag{2.58}$$

where the last inequality is due to (2.54). Also, since $u_n(0) = P_n u_0 \to u_0$ in H^1, then

$$\int_{\mathbb{T}^3} |u_n(0) - u_0|^4 |w(t)|^2 dx \le \|u_n(0) - u_0\|_6^4 \|w(t)\|_6^2$$
$$\le C\|u_n(0) - u_0\|_{H^1}^4 \|w(t)\|_{H^1}^2 \le \epsilon \|w(t)\|_{H^1}^2, \quad \text{for } n \ge N_\epsilon. \tag{2.59}$$

Notice

$$\|w(t)\|_2^2 = \int_{\mathbb{T}^3} |w(t)|^2 dx \le \int_{\mathbb{T}^3} \left| \int_0^t w'(\tau) d\tau \right|^2 dx + \|w(0)\|_2^2$$
$$\le t \int_0^t \|w'(\tau)\|_2^2 d\tau + \|w(0)\|_2^2. \tag{2.60}$$

We let φ be a periodic smooth function such that $\|u_0 - \varphi\|_{H^1}^4 \le \epsilon$. Since φ is smooth, there exists $C_\epsilon > 0$ such that $|\varphi(x)| \le C_\epsilon$ for all $x \in \mathbb{T}^3$. As a result,

$$\int_{\mathbb{T}^3} |u_0|^4 |w(t)|^2 dx \le C \int_{\mathbb{T}^3} |u_0 - \varphi|^4 |w(t)|^2 dx + C \int_{\mathbb{T}^3} |\varphi|^4 |w(t)|^2 dx$$
$$\le C\|u_0 - \varphi\|_6^4 \|w(t)\|_6^2 + C_\epsilon \|w(t)\|_2^2$$
$$\le C\epsilon \|w(t)\|_{H^1}^2 + C_\epsilon t \int_0^t \|w'(\tau)\|_2^2 d\tau + C_\epsilon \|w(0)\|_2^2, \quad \text{for all } t \in [0,T], \tag{2.61}$$

owing to (2.60).

Applying estimates (2.58), (2.59) and (2.61) to the right-hand side of (2.57) yields

$$\int_{\mathbb{T}^3} |u_n(t)|^4 |w(t)|^2 dx \le C(K)(t^{\frac{10}{3}} + \epsilon)\|w(t)\|_{H^1}^2 + C_\epsilon t \int_0^t \|w'(\tau)\|_2^2 d\tau + C_\epsilon \|w(0)\|_2^2, \tag{2.62}$$

for all $t \in [0,T]$, and for all $n \ge N_\epsilon$. Thus, $\int_{\mathbb{T}^3} \left(|u_n(t)|^4 + |u_j(t)|^4\right) |w(t)|^2 dx$ is also bounded by the right-hand side of (2.62) if $n, j \ge N_\epsilon$.

By virtue of (2.53), (2.55), (2.56) and (2.62), we obtain

$$\left| \int_0^t \int_{\mathbb{T}^3} \left(u_n^5 - u_j^5\right) w' dx d\tau \right|$$
$$\le C(K)(t^{\frac{10}{3}} + \epsilon)\|w(t)\|_{H^1}^2 + C_\epsilon t \int_0^t \|w'(\tau)\|_2^2 d\tau$$
$$+ C(K) \int_0^t \left(\|u_n'\|_6 + \|u_j'\|_6\right) \|w\|_{H^1}^2 d\tau + C(K,\epsilon)\|w(0)\|_{H^1}^2, \tag{2.63}$$

for all $t \in [0,T]$ and $n, j \ge N_\epsilon$.

Applying estimates (2.60) and (2.63) to inequality (2.36) with $m = p = 5$, and using $\|w\|_{H^1}^2 = \|\nabla w\|_2^2 + \|w\|_2^2$, we have

$$0 \le \frac{1}{2} \left(\|w(t)\|_{H^1}^2 + \|w'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left[(u_n')^5 - (u_j')^5\right] w' dx d\tau$$
$$\le C(K,\epsilon)\|w(0)\|_{H^1}^2 + \frac{1}{2}\|w'(0)\|_2^2 + C(K)(t^{\frac{10}{3}} + \epsilon)\|w(t)\|_{H^1}^2$$

$$+ C_\epsilon t \int_0^t \|w'(\tau)\|_2^2 d\tau + C(K) \int_0^t \left(\|u_n'\|_6 + \|u_j'\|_6\right) \|w\|_{H^1}^2 d\tau$$

$$+ \left| \int_0^t \int_{\mathbb{T}^3} \left[(P_n - P_j)u_j^5\right] w' dx d\tau \right| + \left| \int_0^t \int_{\mathbb{T}^3} \left[(P_n - P_j)(u_j')^5\right] w' dx d\tau \right|, \tag{2.64}$$

for all $t \in [0, T]$ and $n, j \geq N_\epsilon$.

We remark that our strategy is to first prove the local existence of weak solutions on $[0, T]$, and extend the local solution to a global solution later. Thus, we can choose ϵ and T sufficiently small such that $C(K)(T^{\frac{10}{3}} + \epsilon) \leq \frac{1}{4}$, then (2.64) shows

$$0 \leq \frac{1}{4} \left(\|w(t)\|_{H^1}^2 + \|w'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left[(u_n')^5 - (u_j')^5\right] w' dx d\tau$$

$$\leq C(K, \epsilon)\|w(0)\|_{H^1}^2 + \frac{1}{2}\|w'(0)\|_2^2 + C_\epsilon t \int_0^t \|w'(\tau)\|_2^2 d\tau + C(K) \int_0^t \left(\|u_n'\|_6 + \|u_j'\|_6\right) \|w\|_{H^1}^2 d\tau$$

$$+ \left| \int_0^t \int_{\mathbb{T}^3} \left[(P_n - P_j)(u_j^5)\right] w' dx d\tau \right| + \left| \int_0^t \int_{\mathbb{T}^3} \left[(P_n - P_j)(u_j')^5\right] w' dx d\tau \right|, \tag{2.65}$$

for all $t \in [0, T]$ and $n, j \geq N_\epsilon$. Recall, if one restricts $T \leq 1$, then the uniform bound K defined in (2.54) depends only on $\mathscr{E}(0)$. Thus, T depends only on $\mathscr{E}(0)$.

Because of (2.54), we can apply the Grönwall's inequality to (2.65), it follows that

$$0 \leq \frac{1}{4} \left(\|w(t)\|_{H^1}^2 + \|w'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} \left[(u_n')^5 - (u_j')^5\right] w' dx d\tau$$

$$\leq C(K, T, \epsilon)\Big(\|w(0)\|_{H^1}^2 + \|w'(0)\|_2^2 + \left| \int_0^t \int_{\mathbb{T}^3} [(P_n - P_j)u_j^5] w' dx d\tau \right|$$

$$+ \left| \int_0^t \int_{\mathbb{T}^3} [(P_n - P_j)(u_j')^5] w' dx d\tau \right| \Big), \tag{2.66}$$

for $t \in [0, T]$ and $n, j \geq N_\epsilon$.

Since $w = u_n - u_j$, one has

$$\lim_{n,j\to\infty} (\|w(0)\|_{H^1}^2 + \|w'(0)\|_2^2) = \lim_{n,j\to\infty} (\|P_n u_0 - P_j u_0\|_{H^1}^2 + \|P_n u_1 - P_j u_1\|_2^2) = 0.$$

Then, by using (2.46) and (2.47), we derive from (2.66) that

$$\lim_{j\to\infty} \overline{\lim_{n\to\infty}} \int_0^t \int_{\mathbb{T}^3} \left[(u_n')^5 - (u_j')^5\right] w' dx d\tau = 0, \quad \text{for all } t \in [0, T],$$

under the scenario that $m = p = 5$.

In sum, for both Case A (i.e. $1 \leq p < \frac{5}{6}(m+1)$) and Case B (i.e. $m = p = 5$), we have

$$\lim_{j\to\infty} \varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} \left(|u_n'|^{m-1}u_n' - |u_j'|^{m-1}u_j\right) w' dx d\tau = 0. \tag{2.67}$$

Remark 2.2. For Case A, the length T of the time interval can be arbitrarily large. However, for Case B, we restrict T to be small which depends on $\mathscr{E}(0)$ only. But this restriction does not affect our intention to prove the local existence of weak solutions on $[0,T]$. Local weak solutions will eventually be extended to global ones in subsection 2.7.

We also remark that the "union" of Case A and Case B in the above proof is same as the "union" of Case 1 and Case 2 in the statement of Theorem 1.2.

2.4.3. Completion of the proof for $|u_n'|^{m-1}u_n' \to |u'|^{m-1}u'$ in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$

We recall from (2.26) and (2.32) that $u_n' \to u'$ weakly in $L^{m+1}(\mathbb{T}^3 \times (0,T))$ and $|u_n'|^{m-1}u_n' \to \psi$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$. Then, by (2.67), we have

$$\begin{aligned}
&\varlimsup_{j\to\infty} \varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n' - |u_j'|^{m-1}u_j')(u_n' - u_j') dx dt \\
&= \varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} |u_n'|^{m+1} dx dt + \varlimsup_{j\to\infty} \int_0^T \int_{\mathbb{T}^3} |u_j'|^{m+1} dx dt \\
&\quad - \lim_{j\to\infty} \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n') u_j' dx dt - \lim_{j\to\infty} \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|u_j'|^{m-1}u_j) u_n' dx dt \\
&= 2 \varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} |u_n'|^{m+1} dx dt - 2 \int_0^T \int_{\mathbb{T}^3} \psi u' dx dt = 0.
\end{aligned}$$

It follows that

$$\varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} |u_n'|^{m+1} dx dt = \int_0^T \int_{\mathbb{T}^3} \psi u' dx dt. \tag{2.68}$$

Since $|s|^{m-1}s$ is a monotone increasing function on $\mathbb{R}$, then for any $v \in L^{m+1}(\mathbb{T}^3 \times (0,T))$, we have

$$\int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n' - |v|^{m-1}v)(u_n' - v) dx dt \geq 0. \tag{2.69}$$

Owing to (2.68) and (2.69), as well as the fact that $u_n' \to u'$ weakly in $L^{m+1}(\mathbb{T}^3 \times (0,T))$ and $|u_n'|^{m-1}u_n' \to \psi$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, we obtain

$$\varlimsup_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n' - |v|^{m-1}v)(u_n' - v) dx dt$$

$$= \overline{\lim_{n\to\infty}} \int_0^T \int_{\mathbb{T}^3} |u_n'|^{m+1} dxdt - \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n')v dxdt$$
$$- \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} (|v|^{m-1}v)u_n' dxdt + \int_0^T \int_{\mathbb{T}^3} |v|^{m+1} dxdt$$
$$= \int_0^T \int_{\mathbb{T}^3} \psi u' dxdt - \int_0^T \int_{\mathbb{T}^3} \psi v dxdt - \int_0^T \int_{\mathbb{T}^3} (|v|^{m-1}v)u' dxdt + \int_0^T \int_{\mathbb{T}^3} |v|^{m+1} dxdt$$
$$= \int_0^T \int_{\mathbb{T}^3} (\psi - |v|^{m-1}v)(u' - v) dxdt \geq 0, \quad \text{for any } v \in L^{m+1}(\mathbb{T}^3 \times (0,T)). \tag{2.70}$$

We claim that $v \mapsto |v|^{m-1}v$ is a maximal monotone operator mapping from $L^{m+1}(\mathbb{T}^3 \times (0,T)) \to L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$. It suffices to show that this operator is both monotone and hemicontinuous (see, e.g., [2]). The monotonicity is obvious. It remains to verify the hemicontinuity. Recall that an operator A mapping from a Banach space to its dual is said to be hemicontinuous if $A(v + \lambda y)$ converges weakly to $A(v)$ as $\lambda \to 0$ for all v, y in this Banach space. Hence, to check the operator $v \mapsto |v|^{m-1}v$ is hemicontinuous from $L^{m+1}(\mathbb{T}^3 \times (0,T)) \to L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, it is enough to verify that, for all v, y, $\eta \in L^{m+1}(\mathbb{T}^3 \times (0,T))$,

$$\lim_{\lambda\to\infty} \int_0^T \int_{\mathbb{T}^3} \left[|v + \lambda y|^{m-1}(v + \lambda y)\right] \eta dxdt = \int_0^T \int_{\mathbb{T}^3} (|v|^{m-1}v)\eta dxdt. \tag{2.71}$$

As a matter of fact, (2.71) follows from Lebesgue's dominated convergence theorem.

Since we have shown the maximal monotonicity of the operator $v \mapsto |v|^{m-1}v$ from $L^{m+1}(\mathbb{T}^3 \times (0,T)) \to L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, it can be concluded from (2.70) that $\psi = |u'|^{m-1}u'$. Namely,

$$|u_n'|^{m-1}u_n' \to |u'|^{m-1}u' \ \text{ weakly in } L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T)). \tag{2.72}$$

2.5. Passage to the limit for the Galerkin system

In this section, we let $n \to \infty$ in the Galerkin approximation system, and aim to show that (u, u') is a weak solution for model (1.2)-(1.3) on $[0,T]$. Let ϕ be a trigonometric polynomial with smooth coefficients, i.e., $\phi = \sum_{\substack{k=(k_1,k_2,k_3)\in\mathbb{Z}^3 \\ |k_1|,|k_2|,|k_3|\leq N}} \hat{\phi}(k,t)e^{ik\cdot x}$ where $\hat{\phi}(k,t)$ is smooth in t. We multiply the Galerkin system (2.4) with ϕ and integrate over $\mathbb{T}^3 \times (0,t)$. Assume n is larger than the degree N of ϕ, then $\int_0^t \int_{\mathbb{T}^3} P_n(|u_n'|^{m-1}u_n')\phi dxd\tau = \int_0^t \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n')\phi dxd\tau$, and $\int_0^t \int_{\mathbb{T}^3} P_n(|u_n|^{p-1}u_n)\phi dxd\tau = \int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n)\phi dxd\tau$. It follows that

$$\int_0^t \int_{\mathbb{T}^3} u_n''\phi dxd\tau + \int_0^t \int_{\mathbb{T}^3} \nabla u_n \cdot \nabla\phi dxd\tau + \int_0^t \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n')\phi dxd\tau$$
$$= \int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n)\phi dxd\tau, \quad \text{for all } \ t \in [0,T]. \tag{2.73}$$

Then, since $|u_n'|^{m-1}u_n' \to |u'|^{m-1}u'$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, $|u_n|^{p-1}u_n \to |u|^{p-1}u$ weakly in $L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$, $u_n \to u$ weakly* in $L^\infty(0,T;H^1(\mathbb{T}^3))$ and $u_n'' \to u''$ weakly* in $L^{\frac{m+1}{m}}(0,T;X')$ where $X = H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)$, we can pass to the limit as $n \to \infty$ in (2.73) to obtain

$$\int_0^t \langle u'', \phi\rangle d\tau + \int_0^t \int_{\mathbb{T}^3} \nabla u \cdot \nabla \phi dx d\tau + \int_0^t \int_{\mathbb{T}^3} (|u'|^{m-1}u')\phi dx d\tau$$
$$= \int_0^t \int_{\mathbb{T}^3} (|u|^{p-1}u)\phi dx d\tau, \quad \text{for all } \ t \in [0,T]. \tag{2.74}$$

After integration by parts in time, we obtain

$$\int_{\mathbb{T}^3} u'(t)\phi(t)dx - \int_{\mathbb{T}^3} u'(0)\phi(0)dx - \int_0^t \int_{\mathbb{T}^3} u'\phi' dx d\tau$$
$$+ \int_0^t \int_{\mathbb{T}^3} \nabla u \cdot \nabla \phi dx d\tau + \int_0^t \int_{\mathbb{T}^3} (|u'|^{m-1}u')\phi dx d\tau = \int_0^t \int_{\mathbb{T}^3} (|u|^{p-1}u)\phi dx d\tau, \tag{2.75}$$

for all $t \in [0,T]$, and for any trigonometric polynomial ϕ with smooth coefficients.

Recall $u \in L^\infty(0,T;H^1(\mathbb{T}^3))$, $u' \in L^\infty(0,T;L^2(\mathbb{T}^3))$, and $|u|^{p-1}u$, $|u'|^{m-1}u' \in L^{\frac{m+1}{m}}(\mathbb{T}^3 \times (0,T))$. Thus, by density, we conclude that (2.75) holds for all $\phi \in C([0,T];H^1(\mathbb{T}^3)) \cap L^{m+1}(\mathbb{T}^3 \times (0,T))$ with $\phi' \in C([0,T];L^2(\mathbb{T}^3))$.

We shall verify the initial condition $u(0) = u_0$ and $u'(0) = u_1$. Indeed, let $\tilde{\phi}$ be a trigonometric polynomial with smooth coefficients such that $\tilde{\phi}(T) = \tilde{\phi}'(T) = 0$. By setting $t = T$ and $\phi = \tilde{\phi}$ in (2.73), then after integration by parts in time, we obtain

$$\int_{\mathbb{T}^3} [(P_n u_0)\tilde{\phi}'(0) - (P_n u_1)\tilde{\phi}(0)]dx + \int_0^T \int_{\mathbb{T}^3} u_n \tilde{\phi}'' dx d\tau + \int_0^T \int_{\mathbb{T}^3} \nabla u_n \cdot \nabla \tilde{\phi} dx d\tau$$
$$+ \int_0^T \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n')\tilde{\phi} dx d\tau = \int_0^T \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n)\tilde{\phi} dx d\tau, \tag{2.76}$$

where we have used $u_n(0) = P_n u_0$ and $u_n'(0) = P_n u_1$.

Letting $n \to \infty$ in (2.76), we have

$$\int_{\mathbb{T}^3} [u_0\tilde{\phi}'(0) - u_1\tilde{\phi}(0)]dx + \int_0^T \int_{\mathbb{T}^3} u \tilde{\phi}'' dx d\tau + \int_0^T \int_{\mathbb{T}^3} \nabla u \cdot \nabla \tilde{\phi} dx d\tau$$
$$+ \int_0^T \int_{\mathbb{T}^3} (|u'|^{m-1}u')\tilde{\phi} dx d\tau = \int_0^T \int_{\mathbb{T}^3} (|u|^{p-1}u)\tilde{\phi} dx d\tau. \tag{2.77}$$

However, by setting $t = T$ and $\phi = \tilde{\phi}$ in (2.74), and after integration by parts, we have

$$\int_{\mathbb{T}^3} [u(0)\tilde{\phi}'(0) - u'(0)\tilde{\phi}(0)]dx + \int_0^T \int_{\mathbb{T}^3} u \tilde{\phi}'' dx d\tau + \int_0^T \int_{\mathbb{T}^3} \nabla u \cdot \nabla \tilde{\phi} dx d\tau$$
$$+ \int_0^T \int_{\mathbb{T}^3} (|u'|^{m-1}u')\tilde{\phi} dx d\tau = \int_0^T \int_{\mathbb{T}^3} (|u|^{p-1}u)\tilde{\phi} dx d\tau. \tag{2.78}$$

Comparing (2.77) and (2.78), we obtain

$$\int_{\mathbb{T}^3} [u_0\tilde{\phi}'(0) - u_1\tilde{\phi}(0)]dx = \int_{\mathbb{T}^3} [u(0)\tilde{\phi}'(0) - u'(0)\tilde{\phi}(0)]dx. \tag{2.79}$$

Since $\tilde{\phi}(0)$ and $\tilde{\phi}'(0)$ are arbitrary trigonometric polynomials, we obtain $u(0) = u_0$ and $u'(0) = u_1$. This completes the proof for the existence of weak solutions on $[0, T]$.

2.6. Energy identity

Let (u, u_t) be a weak solution for (1.2)-(1.3) on $[0, T]$ in the sense of Definition 1.1. We aim to show that the energy identity (1.5) holds for all $t \in [0, T]$.

One may multiply equation (1.2) with u_t, and perform integration by parts to obtain the energy identity (1.5) *formally*. However, u_t is not smooth enough, this formal procedure is not rigorous. To remedy it, we regularize solutions by the projection operator P_n defined in (2.1). We set $\phi = P_n u_t$ in the variational identity (1.4) to get

$$\begin{aligned}
&\frac{1}{2}\left(\|P_n u_t(t)\|_2^2 + \|\nabla P_n u(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} (|u_t|^{m-1}u_t)(P_n u_t)dxd\tau \\
&= \frac{1}{2}\left(\|P_n u_t(0)\|_2^2 + \|\nabla P_n u(0)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} (|u|^{p-1}u)(P_n u_t)dxd\tau,
\end{aligned} \tag{2.80}$$

for all $t \in [0, T]$.

Since $u_t \in L^{m+1}(\mathbb{T}^3 \times (0, T))$, one has $u_t \in L^{m+1}(\mathbb{T}^3)$ for a.e. $t \in [0, T]$. Then by (2.2), $P_n u_t \to u_t$ in $L^{m+1}(\mathbb{T}^3)$ as $n \to \infty$ for a.e. $t \in [0, T]$. Also, due to (2.3), we know $\|P_n u_t\|_{m+1} \leq c_m \|u_t\|_{m+1}$. Therefore, by the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^t \int_{\mathbb{T}^3} |P_n u_t - u_t|^{m+1} dxd\tau = 0. \tag{2.81}$$

Thus, by Hölder's inequality,

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{T}^3} (|u_t|^{m-1}u_t)(P_n u_t - u_t)dxd\tau \right| \\
&\leq \left(\int_0^t \int_{\mathbb{T}^3} |u_t|^{m+1} dxd\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_{\mathbb{T}^3} |P_n u_t - u_t|^{m+1} dxd\tau \right)^{\frac{1}{m+1}} \to 0, \text{ as } n \to \infty,
\end{aligned}$$

where the convergence to zero is due to (2.81). It follows that

$$\lim_{n\to\infty} \int_0^t \int_{\mathbb{T}^3} (|u_t|^{m-1}u_t)(P_n u_t)dxd\tau = \int_0^t \int_{\mathbb{T}^3} |u_t|^{m+1} dxd\tau, \text{ for all } t \in [0, T]. \tag{2.82}$$

Analogously, we can derive

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{T}^3}(|u|^{p-1}u)(P_n u_t)dxd\tau = \int_0^t\int_{\mathbb{T}^3}|u|^{p-1}uu_t dxd\tau, \text{ for all } t\in[0,T]. \tag{2.83}$$

Thanks to (2.82) and (2.83), we let $n\to\infty$ in (2.80) to obtain the energy identity

$$\begin{aligned}&\frac{1}{2}\left(\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2\right)+\int_0^t\int_{\mathbb{T}^3}|u_t|^{m+1}dxd\tau\\ &=\frac{1}{2}\left(\|u_t(0)\|_2^2+\|\nabla u(0)\|_2^2\right)+\int_0^t\int_{\mathbb{T}^3}|u|^{p-1}uu_t dxd\tau, \text{ for all } t\in[0,T].\end{aligned} \tag{2.84}$$

Since $\int_0^t\int_{\mathbb{T}^3}|u|^{p-1}uu_t dxd\tau = \frac{1}{p+1}\left(\|u(t)\|_{p+1}^{p+1}-\|u(0)\|_{p+1}^{p+1}\right)$, the energy identity (2.84) can be written in the form of (1.5) stated in Theorem 1.2.

In addition, from (2.84) and performing the same calculation as in (2.6)-(2.12), we can derive

$$\mathscr{E}(t)+\frac{1}{2}\int_0^t\|u_t(\tau)\|_{m+1}^{m+1}d\tau \le (\mathscr{E}(0)+t)\,e^{Ct}, \text{ for all } t\in[0,T], \tag{2.85}$$

where $\mathscr{E}(t)=\frac{1}{2}\left(\|\nabla u(t)\|_2^2+\|u_t(t)\|_2^2\right)+\frac{1}{m+1}\|u(t)\|_{m+1}^{m+1}$, and the constant C in (2.85) is independent of T.

2.7. Extension to global solutions

Given initial data $u_0\in H^1(\mathbb{T}^3)\cap L^{m+1}(\mathbb{T}^3)$ and $u_1\in L^2(\mathbb{T}^3)$, using the Galerkin method, we have proved the existence of a weak solution (u,u_t) for (1.2)-(1.3) on $[0,T]$. By Remark 2.2, T can be arbitrarily large for Case A (i.e., $1\le p<\frac{5}{6}(m+1)$), thus the solution can be extended to a global solution on $[0,\infty)$. However, for Case B (i.e., $m=p=5$), T is small depending only on $\mathscr{E}(0)$, then in this case we extend the local solution to a global solution by using a standard continuation argument given below. Indeed, we assume to the contrary that there exists $T_{max}<\infty$ for which this local solution on $[0,T]$ can not be extended beyond T_{max}. Notice, the energy bound (2.85) holds for all $t\in[0,T_{max})$, namely,

$$\mathscr{E}(t)+\frac{1}{2}\int_0^t\|u_t(\tau)\|_{m+1}^{m+1}d\tau \le M := (\mathscr{E}(0)+T_{max})\,e^{CT_{max}}, \text{ for all } t\in[0,T_{max}). \tag{2.86}$$

Recall T depends only on $\mathscr{E}(0)\le M$. Then consider the same initial value problem with initial data at $t=T$ given by $(u(T),u_t(T))$, and since $\mathscr{E}(T)\le M$ due to (2.86), there exists a solution on $[T,2T]$. Together with the solution on $[0,T]$, this defines a solution on $[0,2T]$. After finitely many times iterations, the solution is extended beyond $T_{max}<\infty$, which leads to a contradiction. This completes the proof for the global existence of weak solutions.

3. Uniqueness of weak solutions and continuous dependence on initial data

This section is devoted to proving Theorem 1.3, namely, the uniqueness of weak solutions as well as the continuous dependence on initial data. We present the proof for the continuous dependence on initial data. The uniqueness of weak solutions follows immediately.

Suppose $(u_0, u_1) \in \big(H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)\big) \times L^2(\mathbb{T}^3)$. Let (u_0^n, u_1^n) be a sequence of periodic functions in $\big(H^1(\mathbb{T}^3) \cap L^{m+1}(\mathbb{T}^3)\big) \times L^2(\mathbb{T}^3)$ such that $\lim_{n\to\infty} \|u_0^n - u_0\|_{H^1} = 0$, $\lim_{n\to\infty} \|u_0^n - u_0\|_{m+1} = 0$, and $\lim_{n\to\infty} \|u_1^n - u_1\|_2 = 0$. In the previous section, we have proved the existence of global weak solutions for system (1.2)-(1.3). Thus, for any $T > 0$, there exists a weak solution (u, u') for system (1.2)-(1.3) on $[0, T]$ with initial data (u_0, u_1). Also, for each $n \in \mathbb{N}$, there exists a weak solution (u_n, u_n') for (1.2)-(1.3) on $[0, T]$ with initial data (u_0^n, u_1^n). We aim to show that (u_n, u_n') converges to (u, u') in the sense of (1.7).

By (1.6) in Theorem 1.2, we have

$$\frac{1}{2}\left(\|\nabla u_n(t)\|_2^2 + \|u_n'(t)\|_2^2\right) + \frac{1}{m+1}\|u_n(t)\|_{m+1}^{m+1} + \frac{1}{2}\int_0^t \|u_n'(\tau)\|_{m+1}^{m+1} d\tau$$
$$\le \left(\frac{1}{2}(\|\nabla u_0^n\|_2^2 + \|u_1^n\|_2^2) + \frac{1}{m+1}\|u_0^n\|_{m+1}^{m+1} + t\right) e^{Ct}, \quad \text{for all } t \in [0, T]. \tag{3.1}$$

Also, one has

$$\frac{1}{2}\left(\|\nabla u(t)\|_2^2 + \|u'(t)\|_2^2\right) + \frac{1}{m+1}\|u(t)\|_{m+1}^{m+1} + \frac{1}{2}\int_0^t \|u'(\tau)\|_{m+1}^{m+1} d\tau$$
$$\le \left(\frac{1}{2}(\|\nabla u_0\|_2^2 + \|u_1\|_2^2) + \frac{1}{m+1}\|u_0\|_{m+1}^{m+1} + t\right) e^{Ct}, \quad \text{for all } t \in [0, T]. \tag{3.2}$$

Notice $\|u\|_{H^1}^2 = \|\nabla u\|_2^2 + \|u\|_2^2 \le \|\nabla u\|_2^2 + C\|u\|_{m+1}^{m+1} + 1$ for $m \ge 1$. Then, since $\lim_{n\to\infty} \|u_0^n - u_0\|_{H^1} = 0$, $\lim_{n\to\infty} \|u_0^n - u_0\|_{m+1} = 0$ and $\lim_{n\to\infty} \|u_1^n - u_1\|_2 = 0$, and on account of (3.1)-(3.2), there exists $K > 0$ such that

$$\|u_n(t)\|_{H^1}^2 + \|u_n'(t)\|_2^2 + \|u_n(t)\|_{m+1}^{m+1} + \int_0^t \|u_n'(\tau)\|_{m+1}^{m+1} d\tau$$
$$+ \|u(t)\|_{H^1}^2 + \|u'(t)\|_2^2 + \|u(t)\|_{m+1}^{m+1} + \int_0^t \|u'(\tau)\|_{m+1}^{m+1} d\tau \le K, \tag{3.3}$$

for all $t \in [0, T]$, for all $n \in \mathbb{N}$.

Denote $y_n = u_n - u$. Then, for all $t \in [0, T]$,

$$\frac{1}{2}\left(\|\nabla y_n(t)\|_2^2 + \|y_n'(t)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} (|u_n'|^{m-1}u_n' - |u'|^{m-1}u')y_n' dx d\tau$$
$$= \frac{1}{2}\left(\|\nabla y_n(0)\|_2^2 + \|y_n'(0)\|_2^2\right) + \int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n - |u|^{p-1}u)y_n' dx d\tau. \tag{3.4}$$

In fact, (3.4) can be established rigorously by employing the regularization procedure used in the proof of the energy identity in subsection 2.6.

Since

$$\frac{1}{m+1}\left(\|y_n(t)\|_{m+1}^{m+1} - \|y_n(0)\|_{m+1}^{m+1}\right) = \int_0^t \int_{\mathbb{T}^3} |y_n(\tau)|^{m-1} y_n(\tau) y_n'(\tau) dx d\tau,$$

then (3.4) can be written as

$$\frac{1}{2}\left(\|\nabla y_n(t)\|_2^2+\|y_n'(t)\|_2^2\right)+\frac{1}{m+1}\|y_n(t)\|_{m+1}^{m+1}+\int_0^t\int_{\mathbb{T}^3}(|u_n'|^{m-1}u_n'-|u'|^{m-1}u')y_n'dxd\tau$$
$$=\frac{1}{2}\left(\|\nabla y_n(0)\|_2^2+\|y_n'(0)\|_2^2\right)+\frac{1}{m+1}\|y_n(0)\|_{m+1}^{m+1}$$
$$+\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau+\int_0^t\int_{\mathbb{T}^3}|y_n|^{m-1}y_ny_n'dxd\tau. \tag{3.5}$$

Note, there exists a constant $c_0>0$ such that $(|a|^{m-1}a-|b|^{m-1}b)(a-b)\ge c_0|a-b|^{m+1}$ for all $a,b\in\mathbb{R}$. Then, since $y_n=u_n-u$, we have

$$\int_0^t\int_{\mathbb{T}^3}(|u_n'|^{m-1}u_n'-|u'|^{m-1}u')y_n'dxd\tau\ge c_0\int_0^t\int_{\mathbb{T}^3}|y_n'|^{m+1}dxd\tau. \tag{3.6}$$

Also, by the Hölder's inequality and Young's inequality, one has

$$\int_0^t\int_{\mathbb{T}^3}|y_n|^{m-1}y_ny_n'dxd\tau\le c_0\int_0^t\int_{\mathbb{T}^3}|y_n'|^{m+1}dxd\tau+C\int_0^t\int_{\mathbb{T}^3}|y_n|^{m+1}dxd\tau. \tag{3.7}$$

Applying (3.6)-(3.7) to equality (3.5) yields

$$\frac{1}{2}\left(\|\nabla y_n(t)\|_2^2+\|y_n'(t)\|_2^2\right)+\frac{1}{m+1}\|y_n(t)\|_{m+1}^{m+1}$$
$$=\frac{1}{2}\left(\|\nabla y_n(0)\|_2^2+\|y_n'(0)\|_2^2\right)+\frac{1}{m+1}\|y_n(0)\|_{m+1}^{m+1}$$
$$+\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau+C\int_0^t\|y_n(\tau)\|_{m+1}^{m+1}d\tau. \tag{3.8}$$

In the following, we estimate the integral $\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau$.

For $1<p\le 3$, by using Hölder's inequality and the imbedding $H^1\hookrightarrow L^6$, we have

$$\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau\le C\int_0^t\int_{\mathbb{T}^3}|y_n|(|u_n|^{p-1}+|u|^{p-1})|y_n'|dxd\tau$$
$$\le C\int_0^t\|y_n\|_6\left(\|u_n\|_{3(p-1)}^{p-1}+\|u\|_{3(p-1)}^{p-1}\right)\|y_n'\|_2d\tau\le C\int_0^t\|y_n\|_6(\|u_n\|_6^{p-1}+\|u\|_6^{p-1})\|y_n'\|_2d\tau$$
$$\le C\int_0^t(\|u_n\|_{H^1}^{p-1}+\|u\|_{H^1}^{p-1})(\|y_n\|_{H^1}^2+\|y_n'\|_2^2)d\tau\le C(K)\int_0^t(\|y_n\|_{H^1}^2+\|y_n'\|_2^2)d\tau, \tag{3.9}$$

for all $t\in[0,T]$ and $n\in\mathbb{N}$, where the last inequality is owing to (3.3).

Next we consider the "supercritical" case $p > 3$. Under such scenario, we apply integration by parts in time to convert y_n' to y_n in the integral $\int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n - |u|^{p-1}u) y_n' dx d\tau$. This idea originates from [6] by Bociu and Lasiecka. Similar calculations have been performed in subsection 2.4.2 in the proof of existence of weak solutions. Indeed, $|u_n|^{p-1}u_n - |u|^{p-1}u = (u_n - u)\xi = y_n \xi$ where $|\xi| \le C(|u_n|^{p-1} + |u|^{p-1})$ and $|\xi'| \le C(|u_n|^{p-2} + |u|^{p-2})(|u_n'| + |u'|)$. Hence, by using integration by parts, we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-1}u_n - |u|^{p-1}u) y_n' dx d\tau \\
&= \int_0^t \int_{\mathbb{T}^3} \xi y_n y_n' dx d\tau = \left[\frac{1}{2} \int_{\mathbb{T}^3} \xi y_n^2 dx \right]_0^t - \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} \xi' y_n^2 dx d\tau \\
&\le C \int_{\mathbb{T}^3} \left(|u_n(t)|^{p-1} + |u(t)|^{p-1} \right) |y_n(t)|^2 dx + C \int_{\mathbb{T}^3} \left(|u_n(0)|^{p-1} + |u(0)|^{p-1} \right) |y_n(0)|^2 dx \\
&\qquad + C \int_0^t \int_{\mathbb{T}^3} \left(|u_n|^{p-2} + |u|^{p-2} \right) (|u_n'| + |u'|) y_n^2 dx d\tau, \ \text{ for all } t \in [0, T].
\end{aligned} \tag{3.10}$$

We estimate each term on the right-hand side of (3.10) as follows.

From Remark 1.4, we know that Case I and Case II in the assumption of Theorem 1.3 can be combined as $p \le \min\{\frac{2}{3}m + \frac{5}{3}, m\}$. Thus $\frac{3(p-1)}{2} \le m + 1$. Then, using Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{T}^3} (|u_n(0)|^{p-1} + |u(0)|^{p-1}) |y_n(0)|^2 dx \le \left(\|u_n(0)\|^{p-1}_{\frac{3(p-1)}{2}} + \|u(0)\|^{p-1}_{\frac{3(p-1)}{2}} \right) \|y_n(0)\|_6^2 \\
&\le C \left(\|u_n(0)\|^{p-1}_{m+1} + \|u(0)\|^{p-1}_{m+1} \right) \|y_n(0)\|^2_{H^1} \le C(K) \|y_n(0)\|^2_{H^1},
\end{aligned} \tag{3.11}$$

due to (3.3).

Furthermore, since $p \le \frac{2}{3}m + \frac{5}{3}$, then $(p-2)\frac{3m+3}{2m-1} \le m + 1$. Therefore,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} (|u_n|^{p-2} + |u|^{p-2})(|u_n'| + |u'|) y_n^2 dx d\tau \\
&\le C \int_0^t \left(\|u_n\|^{p-2}_{(p-2)\frac{3m+3}{2m-1}} + \|u\|^{p-2}_{(p-2)\frac{3m+3}{2m-1}} \right) \left(\|u_n'\|_{m+1} + \|u'\|_{m+1} \right) \|y_n\|_6^2 d\tau \\
&\le C \int_0^t \left(\|u_n\|^{p-2}_{m+1} + \|u\|^{p-2}_{m+1} \right) \left(\|u_n'\|_{m+1} + \|u'\|_{m+1} \right) \|y_n\|^2_{H^1} d\tau \\
&\le C(K) \int_0^t \left(\|u_n'\|_{m+1} + \|u'\|_{m+1} \right) \|y_n\|^2_{H^1} d\tau, \ \text{ for all } t \in [0, T],
\end{aligned} \tag{3.12}$$

where we use (3.3) to obtain the last inequality.

Finally, we consider $\int_{\mathbb{T}^3} (|u_n(t)|^{p-1} + |u(t)|^{p-1}) |y_n(t)|^2 dx$. We write out the estimate for $\int_{\mathbb{T}^3} |u_n(t)|^{p-1} \times |y_n(t)|^2 dx$ only. The estimate for $\int_{\mathbb{T}^3} |u(t)|^{p-1} |y_n(t)|^2 dx$ is similar. Notice

$$\int_{\mathbb{T}^3} |u_n(t)|^{p-1}|y_n(t)|^2dx \le C\int_{\mathbb{T}^3} |u_n(t)-u_n(0)|^{p-1}|y_n(t)|^2dx$$
$$+C\int_{\mathbb{T}^3} |u_n(0)-u_0|^{p-1}|y_n(t)|^2dx + C\int_{\mathbb{T}^3} |u_0|^{p-1}|y_n(t)|^2dx. \tag{3.13}$$

Since $p \le \frac{2}{3}m+\frac{5}{3}$, then $\frac{3(p-1)}{2(m+1)} \le 1$. Therefore,

$$\int_{\mathbb{T}^3} |u_n(t)-u_n(0)|^{p-1}|y_n(t)|^2dx = \int_{\mathbb{T}^3} \left|\int_0^t u_n'(\tau)d\tau\right|^{p-1} |y_n(t)|^2dx$$
$$\le \left(\int_{\mathbb{T}^3} \left|\int_0^t u_n'(\tau)d\tau\right|^{\frac{3(p-1)}{2}} dx\right)^{2/3} \|y_n(t)\|_6^2$$
$$\le Ct^{\frac{m(p-1)}{m+1}} \left(\int_{\mathbb{T}^3} \left(\int_0^t |u_n'(\tau)|^{m+1}d\tau\right)^{\frac{3(p-1)}{2(m+1)}} dx\right)^{2/3} \|y_n(t)\|_{H^1}^2$$
$$\le Ct^{\frac{m(p-1)}{m+1}} \left(\int_{\mathbb{T}^3}\int_0^t |u_n'(\tau)|^{m+1}d\tau dx\right)^{2/3} \|y_n(t)\|_{H^1}^2 \le C(K)t^{\frac{m(p-1)}{m+1}}\|y_n(t)\|_{H^1}^2, \tag{3.14}$$

for all $t \in [0,T]$, due to (3.3).

Next, we consider the integral $\int_{\mathbb{T}^3} |u_n(0)-u_0|^{p-1}|y_n(t)|^2dx$. Recall $u_n(0) = u_0^n \to u_0$ in $L^{m+1}(\mathbb{T}^3)$. Note, the assumption that $p \le \frac{2}{3}m+\frac{5}{3}$ implies $\frac{3(p-1)}{2} \le m+1$. Then, by Hölder's inequality, one has

$$\int_{\mathbb{T}^3} |u_n(0)-u_0|^{p-1}|y_n(t)|^2dx = \int_{\mathbb{T}^3} |u_0^n-u_0|^{p-1}|y_n(t)|^2dx \le \|u_0^n-u_0\|_{\frac{3(p-1)}{2}}^{p-1}\|y_n(t)\|_6^2$$
$$\le C\|u_0^n-u_0\|_{m+1}^{p-1}\|y_n(t)\|_{H^1}^2 \le \epsilon\|y_n(t)\|_{H^1}^2, \quad \text{for all } t\in[0,T], \tag{3.15}$$

for n sufficiently large.

It remains to estimate the integral $\int_{\mathbb{T}^3} |u_0|^{p-1}|y_n(t)|^2dx$. We notice

$$\|y_n(t)\|_2^2 = \int_{\mathbb{T}^3} |y_n(t)|^2dx = \int_{\mathbb{T}^3} \left(\left|y_n(0)+\int_0^t y_n'(\tau)d\tau\right|^2\right)dx$$
$$\le \left(\|y_n(0)\|_2^2 + t\int_0^t \|y_n'(\tau)\|_2^2\,d\tau\right), \quad \text{for all } t\in[0,T]. \tag{3.16}$$

Recall $\frac{3(p-1)}{2} \le m+1$. Then, since $u_0 \in L^{m+1}(\mathbb{T}^3) \subset L^{\frac{3(p-1)}{2}}(\mathbb{T}^3)$, there exists a smooth periodic function φ such that $\|u_0-\varphi\|_{\frac{3(p-1)}{2}}^{p-1} \le \epsilon$. Furthermore, since φ is smooth on $\mathbb{T}^3$, there exists $C_\epsilon > 0$ with $|\varphi(x)| \le C_\epsilon$ for all $x \in \mathbb{T}^3$. It follows that

$$\int_{\mathbb{T}^3} |u_0|^{p-1}|y_n(t)|^2dx \le C\int_{\mathbb{T}^3} |u_0-\varphi|^{p-1}|y_n(t)|^2dx + C\int_{\mathbb{T}^3} |\varphi|^{p-1}|y_n(t)|^2dx$$

$$\leq C\|u_0-\varphi\|^{p-1}_{\frac{3(p-1)}{2}}\|y_n(t)\|_6^2+C_\epsilon\|y_n(t)\|_2^2$$
$$\leq C\epsilon\|y_n(t)\|^2_{H^1}+C_\epsilon\left(\|y_n(0)\|_2^2+t\int_0^t\|y_n'(\tau)\|_2^2\,d\tau\right),\quad \text{for all } t\in[0,T], \tag{3.17}$$

by virtue of (3.16).

By substituting (3.14), (3.15) and (3.17) into (3.13), we obtain, for n sufficiently large,

$$\int_{\mathbb{T}^3}|u_n(t)|^{p-1}|y_n(t)|^2dx$$
$$\leq\left(C(K)t^{\frac{m(p-1)}{m+1}}+C\epsilon\right)\|y_n(t)\|^2_{H^1}+C_\epsilon\left(\|y_n(0)\|_2^2+t\int_0^t\|y_n'(\tau)\|_2^2\,d\tau\right), \tag{3.18}$$

for all $t\in[0,T]$. Using a similar calculation, $\int_{\mathbb{T}^3}|u(t)|^{p-1}|y_n(t)|^2dx$ has the same bound as (3.18).

Now, applying the estimates (3.11), (3.12) and (3.18) to (3.10), it follows that

$$\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau\leq C(K,\epsilon)\|y_n(0)\|^2_{H^1}+\left(C(K)t^{\frac{m(p-1)}{m+1}}+C\epsilon\right)\|y_n(t)\|^2_{H^1}$$
$$+C(K)\int_0^t\left(\|u_n'(\tau)\|_{m+1}+\|u'(\tau)\|_{m+1}\right)\|y_n(\tau)\|^2_{H^1}d\tau+C_\epsilon t\int_0^t\|y_n'(\tau)\|_2^2d\tau, \tag{3.19}$$

for all $t\in[0,T]$ and for sufficiently large n, provided $p>3$.

We have finished estimating the integral $\int_0^t\int_{\mathbb{T}^3}(|u_n|^{p-1}u_n-|u|^{p-1}u)y_n'dxd\tau$ for both of the subcritical case ($1\leq p\leq 3$) and the supercritical case ($p>3$). Then, since $\|y_n\|^2_{H^1}=\|\nabla y_n\|_2^2+\|y_n\|_2^2$, and using estimate (3.9), (3.16) and (3.19), we obtain from (3.8) that

$$\frac{1}{2}\left(\|y_n(t)\|^2_{H^1}+\|y_n'(t)\|_2^2\right)+\frac{1}{m+1}\|y_n(t)\|^{m+1}_{m+1}$$
$$\leq C(K,\epsilon)\|y_n(0)\|^2_{H^1}+\frac{1}{2}\|y_n'(0)\|_2^2+\frac{1}{m+1}\|y_n(0)\|^{m+1}_{m+1}+\left(C(K)t^{\frac{m(p-1)}{m+1}}+C\epsilon\right)\|y_n(t)\|^2_{H^1}$$
$$+C(K)\int_0^t\left(\|u_n'(\tau)\|_{m+1}+\|u'(\tau)\|_{m+1}+1\right)\|y_n(\tau)\|^2_{H^1}d\tau$$
$$+C(T,K,\epsilon)\int_0^t\|y_n'(\tau)\|_2^2d\tau+C\int_0^t\|y_n(\tau)\|^{m+1}_{m+1}d\tau, \tag{3.20}$$

for all $t\in[0,T]$.

Then, by choosing $T_0\in(0,T]$ and $\epsilon>0$ sufficiently small such that $C(K)T_0^{\frac{m(p-1)}{m+1}}+C\epsilon\leq\frac{1}{4}$, we obtain from (3.20) that, for all $t\in[0,T_0]$,

$$\frac{1}{4}\left(\|y_n(t)\|^2_{H^1}+\|y_n'(t)\|_2^2\right)+\frac{1}{m+1}\|y_n(t)\|^{m+1}_{m+1}$$
$$\leq C(K,\epsilon)\|y_n(0)\|^2_{H^1}+\frac{1}{2}\|y_n'(0)\|_2^2+\frac{1}{m+1}\|y_n(0)\|^{m+1}_{m+1}$$

$$+ C(K) \int_0^t \left(\|u_n'(\tau)\|_{m+1} + \|u'(\tau)\|_{m+1} + 1 \right) \|y_n(\tau)\|_{H^1}^2 d\tau$$

$$+ C(T, K, \epsilon) \int_0^t \|y_n'(\tau)\|_2^2 d\tau + C \int_0^t \|y_n(\tau)\|_{m+1}^{m+1} d\tau. \tag{3.21}$$

By virtue of (3.3), we can apply the Grönwall's inequality to (3.21) to conclude

$$\begin{aligned} &\|y_n(t)\|_{H^1}^2 + \|y_n'(t)\|_2^2 + \|y_n(t)\|_{m+1}^{m+1} \\ &\leq C(K, T, \epsilon) \left(\|y_n(0)\|_{H^1}^2 + \|y_n'(0)\|_2^2 + \|y_n(0)\|_{m+1}^{m+1} \right), \ \text{ for all } t \in [0, T_0]. \end{aligned} \tag{3.22}$$

Since

$$\lim_{n\to\infty} (\|y_n(0)\|_{H^1}^2 + \|y_n'(0)\|_2^2 + \|y_n(0)\|_{m+1}^{m+1}) = \lim_{n\to\infty} (\|u_0^n - u_0\|_{H^1}^2 + \|u_1^n - u_1\|_2^2 + \|u_0^n - u_0\|_{m+1}^{m+1}) = 0,$$

then (3.22) implies that $\lim_{n\to\infty} \left[\sup_{t\in[0,T_0]} \left(\|y_n(t)\|_{H^1}^2 + \|y_n'(t)\|_2^2 + \|y_n(t)\|_{m+1}^{m+1} \right) \right] = 0$. By iterating the above procedure for finitely many times, we obtain

$$\lim_{n\to\infty} \left[\sup_{t\in[0,T]} \left(\|y_n(t)\|_{H^1}^2 + \|y_n'(t)\|_2^2 + \|y_n(t)\|_{m+1}^{m+1} \right) \right] = 0.$$

This completes the proof for the continuous dependence on initial data as well as the uniqueness of weak solutions.

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