Sustained Growth with Heterogeneous Households

John H. Boyd III

June 2000

PRELIMINARY DRAFT

Abstract. I examine a model where agents differ in their discount factors. It has long been known that in the long-run, the most patient household or individual ends up owning all the capital, provided there is a maximum sustainable stock. I show that this is no longer true when there is sustained growth. I examine a model where there is endogenous growth due to learning-by-doing, the "Arrow-Romer" model. It is still the case that one household ends up with all the capital, but that household is no longer necessarily the one who has the lowest discount rate. The appropriate measure of impatience depends not just on the discount rate, but also on the form of the felicity function, and the technology. Different technologies can lead to a re-ranking of the households. In some cases, different technologies can even completely reverse the patience ordering, and alter the long-run distribution of capital. There is even the possibility that the most patient household may change depending on the state of development of the economy, causing capital to shift from one household to another as the economy develops.

1. Introduction

It has long been thought that in a competitive economy, the most patient households would end up owning all of the capital. Ramsey (1928) argued that all on the capital would end up in the hands of the most patient households in a competitive equilibrium. The work of Rae (1834) and Fisher (1930) also suggests that the most patient households will accumulate all of the capital. This was confirmed in various frameworks by Rader (1971), Becker (1980) and Bewley (1982). The most patient household does end up with all of the capital.

Epstein and Hynes (1983) examined this issue when households have more general recursive preferences. In a model without borrowing constraints, Epstein and Hynes (1983) note that this extreme capital distribution no longer occurs, provided all households can adjust their rate of impatience to the interest rate. Further examination of the problem shows that when borrowing constraints are present, or when households cannot fully adjust their rate of impatience to the interest rate, the situation becomes more complex (Boyd, 1986). It is true that those who hold capital have lower rates of impatience than those who do not hold capital. However, this ranking does not necessarily correspond to any a priori ranking of households in terms of time preference. A household may be impatient in the steady state either due to intrinsic impatience, or due to poverty. Furthermore, even among those owning capital, the amount owned may not reflect the initial ranking of households in terms of impatience. Capital holdings increase or decrease with a priori impatience depending on whether the rate of impatience is increasing or decreasing.

Arguably, the results with recursive utility are really telling us that ranking households according to impatience may not be straightforward in such circumstances. When we can rank them cleanly, we get results that are broadly similar to the additive cases.

It may come as a surprise to find that even when preferences are additively separable, the most patient household may not end up with all the capital. The presumption that capital flows to the patient depends crucially on the technology involved. When there is a maximum sustainable stock, the conventional wisdom holds, as shown by Rader (1971), Becker (1980), and Bewley (1982). However, this is no longer the case when the economy is capable of sustained growth.

I examine a model where agents differ in their discount factors. This model permits endogenous growth due to learning-by-doing, the "Arrow-Romer" model. It is still the case that one household generally ends up with all the capital, but that household is no longer necessarily the one who has the lowest discount rate. The appropriate measure of impatience depends not just on the discount rate, but also on the form of the felicity function, and the technology. Different technologies can lead to a re-ranking of the households. In some cases, different technologies can even completely reverse the patience ordering, and alter the long-run distribution of capital. There is even the possibility that the most patient household may change depending on the state of development of the economy, causing capital to shift from one household to another as the economy develops.

2. The Ramsey Equilibrium

The Ramsey equilibrium was originally developed by Becker (1980) to address a conjecture of Ramsey (1928). Ramsey argued that, in a competitive economy, the most patient household would end up owning all of the capital. Since this question involves heterogeneous households, the usual additively separable utility function is not terribly well behaved in this setting. The result is that most households eventually sell themselves into slavery (Rader, 1971). One difficulty with this type of model is that it is doubtful that the slaves will want to keep their agreement. It is also hard to see how the "slaves" can continue to provide labor services when they receive absolutely no consumption goods. An alternative approach is to posit incomplete markets. Households cannot borrow against future labor income.[1]

[1] Borrowing constraints that allowed limited borrowing against future wage income would yield similar

The consumer side of model economy consists of H households. Each household's preferences are described by an additively separable utility function with felicity function u_h and discount factor δ_h. Utility is derived solely from consumption of goods. Household h consumes $c_t^h \geq 0$ at time t.

Consumer Preferences. Consumer preferences are described by an additively separable utility function $\sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$ with discount factor δ_h, $0 < \delta_h < 1$. The felicity function $u_h \colon \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable on $\mathbb{R}_{++}$ with $u_h' > 0$, $u_h'' < 0$, and $u_h'(0+) = +\infty$.

We will presume that the discount factors are distinct. For convenience, we label households in order of increasing impatience, so that $1 > \delta_1 > \delta_2 > \cdots > \delta_H > 0$. Household h has labor endowment $L_h > 0$. The total labor endowment of the economy is denoted $L_0 = \sum_{h=1}^{H} L_h$. Let $\ell_h = L_h/L_0$ denote the labor share of household h.

Each household earns income through sale of labor services, and from capital ownership. Because utility is derived solely from consumption goods, the household offers its entire endowment of labor services to the market in each time period. Let w_t denote the wage rate at time t and q_t denote the gross return to capital at time t. The household splits its income at time t into consumption c_t^h and capital holdings x_t^h. Household h starts with an endowment x^h of capital at time zero. Households cannot borrow, which means $x_t^h \geq 0$. The household's budget constraint at time t is $x_t^h + c_t^h \leq q_t x_{t-1}^h + w_t L_h$. Summing up, the household must solve:

The Household's Problem. Given a sequence of wage rates $\{w_t\}$ and gross returns to capital $\{q_t\}$, household h chooses sequences of consumption $\{c_t^h\}$ and capital holding $\{x_t^h\}$

---

results. Moreover, examples 1 and 2 below could be replicated in a model without borrowing constraints.

to solve

$$C^h(w_t, q_t) = \max \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$$

$$\text{s.t. } c_t^h + x_t^h = L_h w_t + q_t x_{t-1}^h, t \geq 1$$

$$c_t^h, x_t^h \geq 0;\ x_0^h = x^h.$$

Firms use an identical, constant returns to scale production function. The technology is described by a constant returns to scale production function $F(K, E)$ where E is effective labor. Effective labor E is subject to Arrovian learning-by-doing, with $E = ZL$, where Z is the total capital stock. As usual, we can simplify matters by assuming there is only one firm, which behaves competitively. In that case, $Z = K$ in equilibrium.

The production sector maximizes profits treating prices and Z_t as given.

The Firm's Problem. The firms must solve a problem of the form

$$P(q_t, w_t, Z_t) = \max \{[F(K_t, Z_t L_t) - q_t K_t - w_t L_t] : K_t, L_t \geq 0\}$$

in every time period.

Of course, equilibrium will require that $K_t = Z_t$. The first-order conditions are $F_1(K_t, Z_t L_t) = q_t$ and $Z_t F_2(K_t, Z_t L_t) = w_t$. Because F is homogeneous of degree one, F_1 and F_2 are homogeneous of degree zero. In equilibrium, $F_1(K_t, K_t L_0) = F_1(1, L_0) = q_t$ and $K_t F_2(K_t, K_t L_0) = K_t F_2(1, L_0) = w_t$. The gross return to capital will be independent of the capital stock (and of time). Thus $q_t = q = F_1(1, L_0)$, while the wage rate will be proportional to the capital stock, with $w_t = K_{t-1} w$ where $w = F_2(1, L_0)$.

In order to state assumptions, it will be useful to define a reduced production function written only in terms of capital. Define $f(K) = F(K, L_0)$. We impose the usual differentiability, concavity, and Inada conditions. Note that the producer's problem is to maximize

$f(k) - qk$, where $k = K/Z$, and that the profits of this firm are the aggregate payments to labor. Thus $f(k) - qk = w\ell$ where $\ell = L_0/Z$.

TECHNOLOGY. The reduced production function is f is twice continuously differentiable on $\mathbb{R}_{++}$ with $f(0) = 0$, $f' > 0$, $f'' < 0$, $f'(0+) = +\infty$, and $f'(+\infty) = 0$.

We are now ready to state the equilibrium conditions.

RAMSEY EQUILIBRIUM. A sequence $\{q_t, w_t, K_t, Z_t, c_t^h, x_t^h\}_{t=1}^{\infty}$ is a Ramsey equilibrium if:

A. c_t^h and x_t^h solve $C^h(w_t, q_t)$ for each h.

B. (K_t, L_0) solves $P(q_t, w_t, Z_t)$ for each t.

C. $\sum_{h=1}^{H} x_{t-1}^h = K_t$ for each t.

D. $K_t = Z_t$ for each t.

The Ramsey equilibrium for this model requires that households maximize utility (A), firms maximize profits (B), capital markets clear (C), and that the aggregate capital stock used to compute effective labor is the actual aggregate capital stock (D). Note that the labor market clearing condition has been incorporated into (B). This is possible due to the constant returns to scale. These imply profits are zero and wages are given by $w_t = F_L(K_{t-1}, L_0) = Z_{t-1}[f(K_{t-1}) - q_t K_{t-1}]/L_0$.

For a wide range of utility and production functions, it is straightforward to modify the existence theorems of Becker, Boyd, and Foias (1991) so that they apply to this model. The key points are to insure that consumer's utility satisfies the appropriate continuity conditions, and to insure that a suitably modified impatience condition is met. Both will be the case in the examples below.

3. Equilibrium Growth

The firm's first-order conditions have been incorporated in our definitions of q and w. The

Inada conditions imposed on the felicity insure that consumption will never be zero, provided the economy started with a non-zero aggregate capital stock. However, households may find that the borrowing constraint binds.

If the borrowing constraint does not bind at time t, the household's first-order condition is the usual Euler equation

$$u_h'(c_t^h) = \delta_h q u_h'(c_{t+1}^h).$$

If the borrowing constraint does bind, marginal utility in period $t+1$ is no higher than in period t, and so

$$u_h'(c_t^h) \geq \delta_h q u_h'(c_{t+1}^h).$$

We move to an analysis of balanced growth in this family of Ramsey equilibrium models. For balanced growth, wages and quantities will grow at a common rate α and the gross return to capital will be constant. More precisely,

Balanced Growth Equilibrium. A Ramsey equilibrium $\{q_t, w_t, K_{t-1}, Z_{t-1}, c_t^h, x_t^h\}_{t=1}^\infty$ exhibits balanced growth if there is an $\alpha > 1$ and $(c^h, x^h, K_0, Z_0, w_1, q)$ so that $c_t^h = \alpha^{t-1}c^h$, $x_t^h = \alpha^t x^h$, $K_t = \alpha^t K_0$, $Z_t = \alpha^t Z_0$, $w_t = \alpha^{t-1}w_1$, and $q_t = q$.

Of course, $q = F_1(1, L_0)$. Since $w_t = K_t w$, we may write $w_t = \alpha^{t-1}w$. Growth in capital stocks at rate α will automatically insure wages also grow at rate α.

To have a balanced growth path, we must have $u_h'(\alpha^t c^h) = \delta_h q u_h'(\alpha^{t+1}c^h)$ for all households who own capital and $u_h'(\alpha^t w1) \geq \delta_h q u_h'(\alpha^{t+1}w_1)$ for all households that do not own capital.

Steady States. The Euler equations tell us that non-trivial steady states can only occur in a knife-edge situation. In a steady state, we have $\delta_h q \leq 1$ for all h. Since the δ_h are ordered, this implies $\delta_h q < 1$ for $h = 2, \ldots, H$. As in Becker (1980), only the most patient household can own capital. Even they will only own capital if $\delta_h q = 1$. If $\delta_h q < 1$, the only steady state is at zero. When $\delta_h q = 1$, household 1 has consumption $(q-1)x^1 + w_1$, and all other households consume w_1.

Before proceeding, we establish a lemma concerning $g_h(c) = u'_h(\alpha c)/u'_h(c)$. Notice that the Euler equations can be written $\delta_h q g(\alpha^t c^h) \leq 1$ when capital is not owned, and $\delta_h q g(\alpha^t c^h) = 1$ when capital is owned. Recall that elasticity of intertemporal substitution (relative risk aversion) is given by $R(c) = -cu''(c)/u'(c)$.

Lemma 1. *Suppose $\alpha > 1$. If R is increasing, $g' < 0$. If R is decreasing, $g' > 0$. If R is constant, $g' = 0$.*

Proof. We calculate $g'(c) = \alpha u''(\alpha c)/u'(c) - u'(\alpha c)u''(c)/[u'(c)]^2$. Rewriting,

$$g'(c) = \frac{u'(\alpha c)}{cu'(c)}\left[\frac{\alpha c u''(\alpha c)}{u'(\alpha c)} - \frac{cu''(c)}{u'(c)}\right] = \frac{u'(\alpha c)}{cu'(c)}[R(c) - R(\alpha c)].$$

Since $\alpha c > c$, increasing R implies a negative g', while decreasing R implies a positive g'. If R is constant, $g' = 0$. $\square$

If g is decreasing or constant (constant or increasing elasticity of intertemporal substitution), $\delta_h q g(c) < \delta_h q(\alpha^{t-1} w_1)$ whenever $c > \alpha^{t-1} w_1$. If a household starts at a zero-capital state, it will stay there. Moreover, if the felicity functions are identical, we can again conclude that only the most patient household will own capital.

More interesting things start to happen if the felicity functions are *not* identical, or if g is increasing. If g is increasing, households will be more likely to own capital as the economy develops. Moreover, this suggests that an impatient household that starts with a large endowment may choose to own capital in periods where a relatively patient household with a small endowment does not choose to own capital.

If the felicity functions differ, there is the possibility that a relatively impatient household will be the only one to own capital. These ideas are developed further in the examples below.

3.1. Examples

Consider a family of simple examples with two households, both with constant elasticities of intertemporal substitution σ_h. We consider technologies of the form $F(K,E) = AK^\gamma E^{1-\gamma}$ with $0 < \gamma < 1$, and presume each household is endowed with one-half unit of labor.

EXAMPLE 1: PATIENT CAPITALIST. Set $A = 4$, $\gamma = 1/2$, $\sigma_1 = 3$, $\sigma_2 = 1$ (log), $\delta_1 = 3/4$, and $\delta_2 = 1/2$. Then $q = F_1(1,1) = \gamma A = 2$, so $\delta_1 q = 3/2$ and $\delta_2 q = 1$. The resulting Euler equations are $(3/2)(c^1_{t+1})^{-3} \le (c^1_t)^{-3}$ and $(c^2_{t+1})^{-1} \le (c^2_t)^{-1}$. With balanced growth at rate α, these become $(3/2)(\alpha^{t+1}c^1)^{-3} \le (\alpha^t c^1)^{-3}$ and $(\alpha^{t+1}c^2)^{-1} \le (\alpha^t c^2)^{-1}$. These reduce to $3/2 \le \alpha^3$ and $1 \le \alpha$. Clearly the second equation cannot bind, so only household 1 owns capital on the balanced growth path. Moreover, the economy has growth factor $(3/2)^{-1/3}$. In this case, the most patient household ends up with the capital.

We now change the technology in Example 1 to make it more productive. Set $A = 16$.

EXAMPLE 2: IMPATIENT CAPITALIST. Now the Euler equations become $6(c^1_{t+1})^{-3} \le (c^1_t)^{-3}$ and $4(c^2_{t+1})^{-1} \le (c^2_t)^{-1}$. With balanced growth at rate α, this reduces to $6 \le \alpha^3$ and $4 \le \alpha$. In this case, $4 \le \alpha$ implies $6 < \alpha^3$, which means that household 1 will own no capital. The impatient household, household 2, is the one that owns capital along the balanced growth path.

In Example 2, the conventional wisdom that the most patient agent ends up with all the capital no longer holds. The most impatient agent actually owns the capital in balanced growth. The reason this phenomenon arises is fairly straightforward. Along growing consumption paths, discounting has two sources. The first source is the discount factor. The second source is the rate of decline in marginal utility of consumption along the growing path. In the case where the elasticity of intertemporal substitution is constant, the effective discounting is $\delta_h \alpha^{-\sigma_h}$. As a result, the growth rate α affects effective discounting, and affects which household owns capital along balanced growth paths. If the elasticities σ_h differ, the ranking of households in terms of effective impatience will depend upon the growth rate.

If we examine the role of the productivity coefficient A, we find that both households will own capital if $(3/8)A = \alpha^3$ and $(1/4)A = \alpha$. I.e., if $A^3/64 = 3A/8$, or $A^2 = 24$. When $A < 2\sqrt{6}$, household one will seem most patient. When $A > \sqrt{6}$, household two will seem most patient.

Of course, other forms of utility will lead to a change in the way the patience ranking works. Suppose $u_h(c) = -e^{-\sigma c}/\sigma$ for $\sigma > 0$. For this household to own capital at a growth rate α, we must have $\delta q e^{-\sigma\alpha c} = e^{-\sigma c}$, or $\alpha = 1 + (1/\delta c)\log \delta_h q$. The growth rate that induces capital ownership depends on the level of current consumption c. As consumption increases, household effectively becomes more patient. Such an economy will not exhibit *balanced* growth, although it will grow. It should be clear that the asymptotic growth rate will be zero.[2]

4. Concluding Remarks

The balanced growth examples show that both technology and preferences can influence the long-run distribution of capital when growth is sustained.

There are still several open questions remaining. First, when they exist, are the balanced growth paths typical of long-run behavior? More precisely, what are the stability properties of Ramsey equilibrium with sustained growth? Second, there is the question of how other types of technologies generating growth might affect long-run income distribution. Do we get similar results if growth is due to human capital accumulation as in Lucas (1988), or to changes in the variety of producer goods (Romer, 1987)? Third, what happens to the long-run income distribution when there is no balanced growth path?

We already have some clues concerning the third question. We can see that there will be no balanced growth path if the elasticity of intertemporal substitution is not constant. When the elasticity is not constant, the growth rate will have to vary, although it may approach a constant rate asymptotically. In this case, capital holdings will affect the effective discount

---

[2] In fact, growth is asymptotically linear in this case.

rate, and it seems likely that the long-run income distribution will depend not only on preferences and technology, but also on the initial capital distribution.

References

Robert A. Becker (1980), On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households, *Quart. J. Econ.* **47**, 375–382.

Robert A. Becker, John H. Boyd III and Ciprian Foias (1991), The Existence of Ramsey Equilibrium, *Econometrica*, 441–460.

Robert A. Becker and Ciprian Foias (1986), A Characterization of Ramsey Equilibrium, *J. Econ. Theory* **41**, 173–183.

Truman Bewley (1982), An Integration of Equilibrium Theory and Turnpike Theory, *J. Math. Econ.* **10**, 233–267.

John H. Boyd III, "Preferences, Technology and Dynamic Equilibria," Ph.D. Dissertation, Indiana University, 1986.

Larry G. Epstein and J. Allan Hynes (1983), The Rate of Time Preference and Dynamic Economic Analysis, *J. Polit. Econ.* **91**, 611–635.

Irving Fisher, "The Theory of Interest," Macmillan, New York, 1930.

Robert E. Lucas, Jr. (1988), On the mechanics of economic development, *J. Mon. Econ.* **22**, 3–43.

Trout Rader, "The Economics of Feudalism," Gordon and Breach, New York, 1971.

John Rae, "Statement of Some New Principles on the Subject of Political Economy," reprinted 1964, Augustus M. Kelly, New York, 1834.

Frank P. Ramsey (1928), A Mathematical Theory of Saving, *Econ. J.* **38**, 543–559.

Paul M. Romer (1987), Growth based on increasing returns due to specialization, *Amer. Econ. Rev.* **98**, 56–62.