Chapter 3 Probability

3.1 Events, sample space, and probability

- **Basic definitions:**

An _______________ is an act of observation that leads to a single outcome that cannot be predicted with certainty.

A ________________ **(or simple event)** is the most basic outcome of an experiment.

A ________________ (denoted S) is the collection of all possible outcomes of an experiment.

An _______________ is a specific collection of sample points.

Examples:

1. **Experiment**: toss a coin and observe the up face.
   Sample points: _______________
   Sample space: ______________
   Event A: head observed _____________

2. **Experiment:** toss a die and observe the up face.
   Sample points: _____________

   Sample space: ______________
   Event A: even # observed _____________
   Event B: # observed at least 3 ________________

3. **Experiment**: toss two coins and observe the up faces.
   Sample space: S: __________________
   Event A: at least one head observed __________________

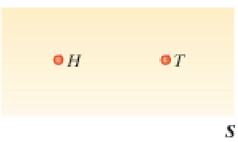
4. **Experiment**: toss three coins and observe the up faces.
   Sample space: S: ____
   Event A: at least one tail observed A: ____

Q: (sample points?): 1. randomly pick up a card from a standard deck of 52 playing cards, the shape on the card is observed.

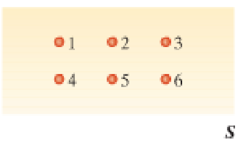
2. randomly pick up a card from a standard deck of 52 playing cards, the number on the card is observed.

3. randomly pick up a ball from a bag with 5 red balls, 8 black balls and 10 green balls, the color is observed.

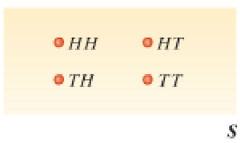
Venn diagram:



a. Experiment: Observe the up face on a coin



b. Experiment: Observe the up face on a die



c. Experiment: Observe the up faces on two coins

- **The _______________ of a ________________:** measures the likelihood that the outcome will occur for a large number repetition.

- **<u>Law of Large Numbers</u>**: As a procedure is repeated over and over again, the relative frequency of an event tends to approach the true probability for that event.

For example, throw a fair coin 10 times and 10000 times. Compare the P(H) and P(T).
10 times, for example, 4 heads, 6 tails, then P(H) = _____ P(T) = ______
10000 times, for example, 4900 heads, 5100 tails, then P(H) = _____ P(T) = _____
So you can understand why we define the true P(H) = P(T) = ________ for a fair coin.

- **Find the probabilities for each sample point.**

Sample space: S: $\{H, T\}$

Sample space: S: $\{1, 2, 3, 4, 5, 6\}$

Sample space: S: $\{HH, HT, TH, TT\}$

- **Probability rules for sample points:**

Let p_i represent the probability of sample point i,

1. All sample point probabilities must lie between 0 and 1: ___________

2. The probabilities of all the sample points within a sample space must sum to 1: ___________

Q: Which of the following assignments of probabilities to the sample points *A*, *B*, and *C* is valid if *A*, *B*, and *C* are the only sample points in the experiment?

$$a: P(A) = \frac{1}{5},\quad P(B) = \frac{1}{8},\quad P(C) = \frac{1}{16}$$
$$b: P(A) = \frac{1}{6},\quad P(B) = \frac{1}{3},\quad P(C) = \frac{1}{2}$$
$$c: P(A) = \frac{1}{6},\quad P(B) = -\frac{1}{3},\quad P(C) = \frac{7}{6}$$

- The _______________**of an** _______________ is calculated by summing the probabilities of the sample points in event A.

Problem A fair die is tossed, and the up face is observed. If the face is even, you win \$1. Otherwise, you lose \$1. What is the probability that you win?

Solution Recall that the sample space for this experiment contains six sample points:

$$S: \{1,2,3,4,5,6\}$$

Event A: toss a die and observe an even number, ______________

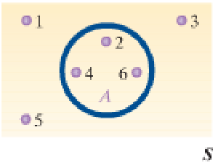


FIGURE 3.4
Die-toss experiment with event A, observe an even number

So P(A) =

So in the long run, you will win \$1 with ________.

Problem Consider the experiment of tossing two *unbalanced* coins. Because the coins are *not* balanced, their outcomes (H or T) are not equiprobable. Suppose the correct probabilities associated with the sample points are given in the accompanying table. [*Note:* The necessary properties for assigning probabilities to sample points are satisfied.] Consider the events

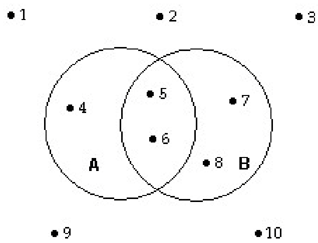
A: {Observe exactly one head.}

B: {Observe at least one head.}

Calculate the probability of A and the probability of B.

Sample Point	Probability
HH	$\frac{4}{9}$
HT	$\frac{2}{9}$
TH	$\frac{2}{9}$
TT	$\frac{1}{9}$

Example1. The accompanying Venn diagram describes the sample space of a particular experiment and events A and B. Suppose $P(1) = P(2) = P(3) = P(4) = 1/16$ and $P(5) = P(6) = P(7) = P(8) = P(9) = P(10) = 1/8$. Find $P(A)$ and $P(B)$.



Example2. 1. List all the possible outcomes when a couple has three children.

S: {____}

2. Find the probability of event A that a couple has three children at least two boys.

Define event A: three children at least two boys

A: {____________________________}

3. Find the probability of event B that a couple has three children at least one girl.

Define event B: three children at least one girl

B: {____}

- **How to determine the number of sample points for an experiment?**

Counting rules:

The <u>**fundamental counting rule**</u> is also called the **multiplication of choices**.

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdot \cdot \cdot k_n$$

Example 1: How many possible outcomes are there for a family having 5 children?

Example 2: A school ID consists of one letter followed by 5 numbers. How many possible ID can be made?

Example3: A paint manufacturer wishes to manufacture several different paints. The categories include

Color: red, blue, white, black, green, brown, yellow

Type: latex, oil

Texture: flat, semigloss, high gloss

Use: outdoor, indoor

How many different kinds of paint can be made if you can select one color, one type, one texture, and one use?

Example 4: A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there? (fall and spring show can't be the same)

- **Combinations rule:**

Combination is a grouping of objects. Order does not matter.

A sample (group) of n elements is to be drawn from a set of N elements. (if order doesn't matter),

Then, the **number** of different possible samples (groups) is denoted by $\binom{N}{n}$ and is equal to

Where the factorial symbol (!) means that

Example1. Consider the task of choosing 2 marines for a dangerous mission from 4 marines (M1, M2, M3, M4). Determine how many different selections can be made.

Q: What is the probability (M2 and M4) combination is to be selected?

Example2. Suppose you wish to select 5 students as a committee from 100 students. How many different groups can you have?

Q: What is the probability you and your 4 best friends as a group to be selected?

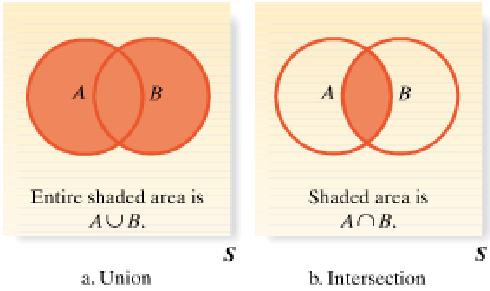
3.2 Unions and intersections

An event can be often viewed as a composition of two or more other events.

The _______ of two events A and B (denoted as A or B,________): the event that occurs if either A or B or both occur on a single performance of the experiment. $A \cup B$ consist of all the sample points that belong to A or B or both.

The __________ of two events A and B (denoted as A and B,__________): the event that occurs if both A and B occur on a single performance of the experiment. $A \cap B$ consist of all the sample points belonging to both A and B.

FIGURE 3.6
Venn diagrams for union and intersection



Example1: Consider a die-toss experiment. Define the following events:

A: {Toss an even number}

B: {Toss a number less than or equal to 3}

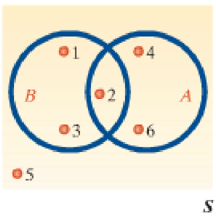


FIGURE 3.7
Venn diagram for die toss

a. Describe $A \cup B$ for this experiment.

b. Describe $A \cap B$ for this experiment.

c. Calculate P($A \cup B$) and P($A \cap B$) assuming the die is fair.

Example2: Continue example 1, define event C: {Toss a number greater than 1},
Find the sample points in

a. Describe $A \cup B \cup C$ for this experiment.

b. Describe $A \cap B \cap C$ for this experiment.

c. Calculate P($A \cup B \cup C$) and P($A \cap B \cap C$) assuming the die is fair.

Example3. Find probability from a two-way table.

Percentage of Respondents in Age-Income Classes

	income		
Age(years)	<$25,000	$25,000-$50,000	>$50,000
< 30	5%	12%	10%
30-50	14%	22%	16%
>50	8%	10%	3%

Define events: A: {A respondent's income is more than $50,000}

B: {A respondent's age is 30 or more}

a. Find P(A) and P(B).

b. Find P($A \cap B$);

c. Find P($A \cup B$).

3.3 Complementary Events

The ___________ of an event A (denoted as______): an event that A does not occur. That is, the event consisting of all sample pointes that are not in event A.

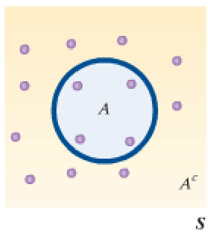


FIGURE 3.8
Venn diagram of complementary events

Rule of Complements:

Note: This rule is useful to calculate the probability of "at least one".

Example1: The weather forecast says there will be 40% to rain today, what the chance it won't rain today?

Example2: Find the probability for a couple having three children at least one is girl.

Example 3: consider the experiment of tossing fair coins.

Define the event A: {observing at least one head}

a. Find P(A) if 2 coins are tossed.
b. Find P(A) if 5 coins are tossed.

3.4 The Additive and Mutually exclusive Events

- The additive rule of probability (find the probability of union)

Example1. For a person selected randomly from a certain population, events A and B are defined as follows: event A: {the person is male}, event B: {the person is a smoker}
It is found that P(A) = 0.51, P(B) = 0.30, and P($A \cap B$) = 0.14, find the probability that a person is selected as a male or a smoker or both?

Example2. Hospital records show that 12% of all patients are admitted for surgical treatment, 16% are admitted for obstetrics, and 2% receive both obstetrics and surgical treatment. If a new patient is admitted to the hospital, what is the probability that the patient will be admitted either for surgery, obstetrics, or both?
define event A: {A patient admitted to the hospital receives surgical treatment}
event B: {A patient admitted to the hospital receives obstetrics treatment}

Example3: One card is randomly selected from 52 standard playing cards, what is the probability it is either a King or a heart card?

Example4: The manager of a used car lot took inventory of the automobiles on his lot and constructed the following table based on the age of his car and its make (foreign or domestic):

Age of Car (in years)

Make	0 – 2	3 – 5	6 – 10	over 10	Total
Foreign	39	26	14	21	100
Domestic	41	29	12	18	100
Total	80	55	26	39	200

1) If a car were randomly selected from the lot, what is the probability that it is either a foreign car or less than 3 years old?

2) If a car were randomly selected from the lot, what is the probability that it is either a domestic car or over 10 years old?

Note: Here is a special relationship between events A and B when $A \cap B$ contains no sample points. We call events A and B __________________________ **events.**

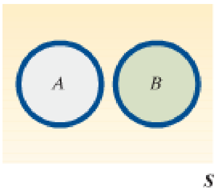


FIGURE 3.11
Venn diagram of mutually exclusive events

Example1: If event A and B are mutually exclusive events, it is found that P(A) = 0.51, P(B) = 0.30, find the probability $P(A \cup B)$?

Example2: Event A and B, P(A) = 0.50, P(B) = 0.30, $P\left(A \cup B\right) = 0.75$, are event A and B mutually exclusive?

Example3: Event A and B, P(A) = 0.50, P(B) = 0.30, $P\left(A \cup B\right) = 0.80$, are event A and B mutually exclusive?

Example4: At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Example5. One card is randomly selected from 52 standard playing cards, what is the probability it is either a King or a 8?

Example6: The manager of a used car lot took inventory of the automobiles on his lot and constructed the following table based on the age of his car and its make (foreign or domestic):

Age of Car (in years)

Make	0 – 2	3 – 5	6 – 10	over 10	Total
Foreign	39	26	14	21	100
Domestic	41	29	12	18	100
Total	80	55	26	39	200

Q1: Which events are mutually exclusive, which are not?

Define event A: {foreign car} event B: {less than 3 years old car}

event C: {domestic car} event D: {more than 10 years old car}

Q2: What's the probability a randomly selected car falls between 3 to 10 years old?

Example4: Exercise 3.42(p134):

Experiment: toss two fair dice, define

Event A: {the sum of the two dice is equal to 7}
Event B: {at least one of the dice is a 4}

a. Identify the sample points in the events A, B, A$\cap$B, A$\cup$B, and A^c.

First list the sample space for this experiment:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

then
Event A:

Event B:

Event A$\cap$B:

Event A$\cup$B:

Event A^c: {	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	
	(2, 1)	(2, 2)	(2, 3)	(2, 4)		(2, 6)
	(3, 1)	(3, 2)	(3, 3)		(3, 5)	(3, 6)
	(4, 1)	(4, 2)		(4, 4)	(4, 5)	(4, 6)
	(5, 1)		(5, 3)	(5, 4)	(5, 5)	(5, 6)
		(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)}

b. Find P(A), P(B), P($A \cap B$), P($A \cup B$) and P(A^c) by summing the probabilities of the appropriate sample points.

c. Use the additive rule to find P($A \cup B$).

d. Are event A and B mutually exclusive? Why?

f. Define event C: {the sum of the two dice is equal to 9}
Are event A and C mutually exclusive? Why?
Are event B and C mutually exclusive? Why?

3.5 Conditional Probability

Q1: There are 52 standard cards. If a card is randomly selected, what is the probability that the card is a King?

Q2: There are 52 standard cards. What is the probability that the second card is selected as a King **given the first card is selected as a King** (without replacement)?

___________ **Probability:** the probability that event A occurs given event B occurs, we divide the probability that both event A and B occur by the probability that B occurs,

We assume that $P(B) \neq 0$.

Example1. Rolling a die, Define

Event A: {3 is observed}, event B: {the die comes up odds}

1. Find the probability P(A), P(B) and $P(A \cap B)$.

2. Find the probability that a 3 is rolled, given the die comes up odd.

Example2: Smoking and cancer:

Many medical researchers have conducted experiments to examine the relationship between cigarette smoking and cancer. The following table shows a certain experiment result.

Define A: {an adult male smokes} B: {an adult male develops cancer}

A^C : {an adult male does not smoke} B^C : {an adult male dose not develop cancer}

then the sample space is:

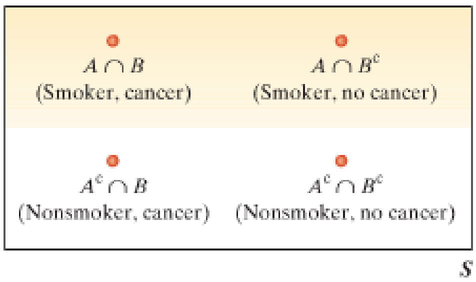


FIGURE 3.14
Sample space for Example 3.15

Probabilities of smoking and developing cancer:

	Develops cancer	
Smoker	Yes (B)	No (B^C)
Yes(A)	0.05	0.20
No(A^C)	0.03	0.72

1. Find the probability that an adult male smokes. P(A)

2. Find the probability that an adult male develops cancer. P(B)

3. Find the probability that an adult male smokes and develops cancer. P($A \cap B$)

4. Find the probability that an adult male develops cancer given he is a smoker.

5. Find the probability that an adult male develops cancer given he is not a smoker.

Example3: The manager of a used car lot took inventory of the automobiles on his lot and constructed the following table based on the age of his car and its make (foreign or domestic):

Age of Car (in years)

Make	0 – 2	3 – 5	6 – 10	over 10	Total
Foreign	39	26	14	21	100
Domestic	41	29	12	18	100
Total	80	55	26	39	200

1) Given a randomly selected car from the car lot is domestic, what is the probability it is over 10 years?
Define event A: {over 10 years old}, B: {domestic car}

2) Given a randomly selected car from the car lot is over 10 years old, what is the probability it is a domestic car?
Define event A: {over 10 years old}, B: {domestic car}

3) Given a randomly selected car from the car lot is over 10 years old, what is the probability it is a foreign car?

4) Given a randomly selected car from the car lot is less than 3 years old, what is the probability it is a foreign car?

Example4: There are 5 red balls, 8 black balls and 10 green balls in a bag. What is the probability that the second ball is selected as a red ball given the first one was a black ball (without replacement)?

Example5: If event A and B are mutually exclusive events, it is found that P(A) = 0.51, P(B) = 0.30, find the probability $P(A|B)$ *and* $P(B|A)$?

3.6 The Multiplicative rule and independent events

- Multiplicative rule of probability (find the probability of intersection)

Example1: There are 52 standard cards. Two cards are selected randomly, **w**hat is the probability that two King cards will be selected (without replacement)?

Example2: There are 5 red balls, 8 black balls and 10 green balls in a bag. Two balls are selected randomly. What is the probability that the first one is selected as a black ball **and** the second ball is a red ball (without replacement)?

Example3: An investor in wheat futures is concerned with the following events:

A: {A serious drought will occur next year}

B: {U.S. production of wheat will be profitable next year}

Based on available information, the investor believes that the probability is 0.01 that production of wheat will be profitable assuming a serious drought will occur in the same year and that the probability is 0.05 that a serious drought will occur. **What is the probability that a serious drought will occur and that a profit will be made?**

- **Independent events:**

--

Problem Consider the experiment of tossing a fair die, and let

A = {Observe an even number.}

B = {Observe a number less than or equal to 4.}

Are A and B independent events?

Q1: Experiment: Toss a coin and roll a die,
Event A: coin comes up head, Event B: die comes up 4.
Are they independent events?

Q2: Pick up 2 cards from a standard 52 standard cards.
Event A: the first card is a King, Event B: the second card is a King (**without replacement**)
Are they independent events?

Q3: Pick up 2 cards from a standard 52 standard cards.
Event A: the first card is a King, Event B: the second card is a King (**with replacement**)
Are they independent events?

Note: If event A and B are _______________, they can't be independent, they are _______________ events.

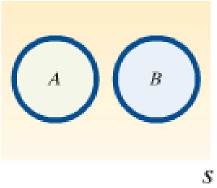


FIGURE 3.19
Mutually exclusive events are dependent events

- ## The Probability of Intersection of Two Independent Events:

____

Example1: Consider the experiment of tossing a fair die and let
A: {Observe an even number}
B: {Observe a number less than or equal to 4}

We already prove that events A and B are independent in the previous example, Find $P(A \cap B)$.

Example2: Consider a couple has two children, assume the chance to have a boy or a girl is equal and independent.

1. Find the probability all two children are girls.

2. Find the probability that there is at least one boy in this family.

Example3: If event A and B are independent events, it is found that P(A) = 0.51, P(B) = 0.30, find the probability $P(A \cap B)$ *and* $P(B \cup A)$?

Example4: Event A and B, P(A) = 0.50, P(B) = 0.30, $P(A \cap B) = 0.15$, are event A and B independent or not?

Example5: Event A and B, P(A) = 0.50, P(B) = 0.30, $P(A \cap B) = 0.21$, are event A and B independent or not?

- **The Probability of Intersection of n independent events:**

____

Example1: Roulette, An American roulette wheel contains 38 numbers, of which 18 are red, 18 are black, and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. In three plays at a roulette wheel, what is the probability that the ball will land on green first time and on black the second and third times?
Define: Event G1: {the ball land on green the first time}
Event B2: {the ball land on black the second time}
Event B3: {the ball land on back the third time}

Example2. A machine has four components, A, B, C, and D, set up in such a manner that all four parts must work for the machine to work properly. Assume the probability of one part working does not depend on the functionality of any of the other parts. Also assume that the probabilities of the individual parts working are P(A) = P(B) = 0.95, P(C) = 0.9, and P(D) = 0.99. Find the probability that the machine works properly.

Example3: There are 5 multiple choice questions in a quiz, each one has four answer options and only one is correct. If you randomly pick up the answer by blind guessing,

1. what is the probability you get all 5 questions correctly?

2. what is the probability that you get at least one question wrong?

3. what is the probability you get all 5 questions wrong?

4. what is the probability that you get at least one question correctly?

Learning Objective for Chapter 3:

1. Identify the sample point, sample space for a experiment and an event
2. Understand the union, intersection of events and complement of an event
3. Calculate the probability for the complementary event
4. Indentify and Calculate the union probability by the additive rule (understand mutually exclusive events and application)
5. Indentify and Calculate the conditional probability
6. Indentify and Calculate the intersection probability by the multiplicative rule (understand independent events and application)