Chapter5 Continuous Random Variables

5.3 The Normal Distribution

One of the most commonly observed continuous random variables has a **bell-shaped probability distribution** (or **bell curve**), as shown in Figure 5.5. It is known as a **normal random variable** and its probability distribution is called a **normal distribution**.

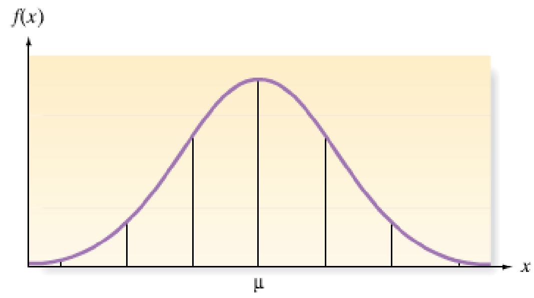


FIGURE 5.5
A normal probability distribution

Probability Distribution for a Normal Random Variable *x*

Probability density function: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$

where

μ = Mean of the normal random variable x
σ = Standard deviation
π = 3.1416...
e = 2.71828...

$P(x < a)$ is obtained from a table of normal probabilities.

- **Probability Distribution for a normal random variable x:**

1. It is _______________ and _____________ about its mean μ .

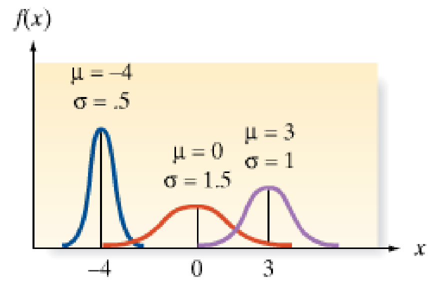
2.___________________________
(the _______ that x falls in the interval a < x < b is the _____ under the curve between the two points a and b.)

3. The Empirical Rule applies to the Normal distribution (graph)
_________ of the data fall in 1 standard deviation within the mean,
_________ of the data fall in 2 standard deviations within the mean,
_________ of the data fall in 3 standard deviations within the mean,
so we know that it is ____ to find data points outside of ± 3 standard deviations from the mean.

The normal distribution plays a very important role in the science of statistical inference.

Many random variables follow a normal or approximately normal distribution, then for different means μ and different standard deviations σ, we could have a large number of normal curves.

FIGURE 5.6
Several normal distributions with different means and standard deviations



To find probability of the different normal random variables, we have formed a single table that will apply to any normal curve.

Property of Normal Distributions

If x is a normal random variable with mean μ and standard deviation σ, then the random variable z defined by the formula

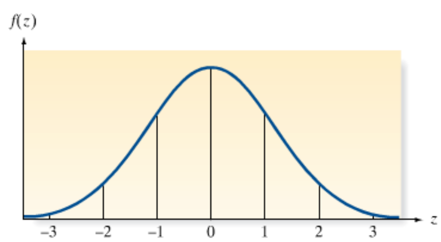
$$z = \frac{x - \mu}{\sigma}$$

has a standard normal distribution. The value z describes the number of standard deviations between x and μ.

The standard normal distribution is a normal distribution with ________ and __________.

A random variable with a standard normal distribution, denoted by the symbol ____, is called a _____________________________________.

FIGURE 5.7
Standard normal distribution: $\mu = 0, \sigma = 1$



- **Properties of standard normal distribution:**

1. The standard normal distribution is ________ about its mean ____.

2. The total area under the standard normal probability distribution equals _____.

3. The _____ under normal curve is equal to the associated _____ under the standard normal curve.

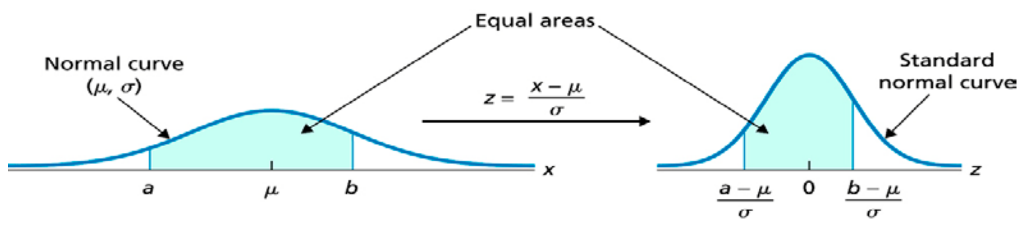
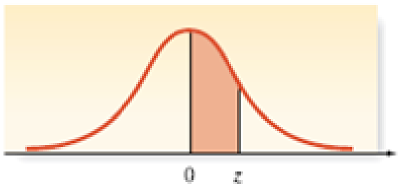


TABLE IV Normal Curve Areas



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Source: Abridged from Table I of A. Hald, *Statistical Tables and Formulas* (New York: Wiley), 1952. Reproduced by permission of A. Hald.

Note: 1. The left column lists the z-value with first decimal, the first row lists the second decimal digit for z-value.

2. The entries in the body of the table give the probability (area) between 0 and z. (shaded area)

Using Z-table (Standard Normal Curve Areas, P794, Table IV) to find the probability:

Examples:

1. $P(z < 0) =$ $\qquad$ $P(z \geq 0) =$

2. $P(0 < z < 0.23) =$ $\qquad$ $P(0 \leq z \leq 1.28) =$

3. $P(-2.33 < z < 0) =$ $\qquad$ $P(-0.67 < z < 0) =$

4. $P(z \geq 1.64) =$ $\qquad$ $P(z \geq 0.85) =$

5. $P(z < -1.46) =$ $\qquad$ $P(z < -0.33) =$

6. $P(z \geq -1.56) =$ $\qquad$ $P(z \geq -0.37) =$

7. $P(z < 1.28) =$ $\qquad$ $P(z < 2.25) =$

8. $P(-2.50 < z < 1.50) =$ $\qquad$ $P(-1.64 < z < 1.28) =$

9. $P(-1 < z < 1) =$ $P(-2 < z < 2) =$ $P(-3 < z < 3) =$

10. $P(0.67 \le z \le 1.96) =$ $P(-2.33 \le z \le -1.05) =$

12. $P(z \ge 4.33) =$ $P(z < 4.33) =$

13. $P(z \le -5) =$ $P(z > -5.0) =$

Work on a normal random variable x with μ *and* σ :

Steps for finding a Probability corresponding to a Normal Random Variable X:

1. Sketch the ____________ (bell curve) and indicate the ______ of the random variable X.
2. Shade the __________ corresponding to the probability of interest.
3. Convert the boundaries (x values) of the shaded area to z values by using ___________.
4. Use the Z – table to find the areas corresponding to the z values.

Example: The random variable x has a normal distribution with $\mu = 70$ and $\sigma = 10$. Find the following probabilities:

1. $P(x \ge 90) =$

2. $P(x < 75) =$

3. $P(60 \le x < 75)$

4. $P(x = 75)$

Example1: Assume that the length of time x, between charges of a cellular phone is normally distributed with a mean of 10 hours and a standard deviation of 1.5 hours.

a. Find the probability that the cell phone will last between 8 and 12 hours between charges.

b. Find the probability that the cell phone will last less than 7 hours.

c. Find the probability that the cell phone will last more than 14 hours.

*Note: We will consider an event to be rare if it has less than a ______ chance of occurring.

Example2. SAT scores are normally distributed with a mean of 998 and a standard deviation of 202 points.

a. Find the probability that a person scores above 1100 on the SAT.

b. Find the probability that a person scores below 700 on the SAT

Example3. Random variable x is the in-city mileage for a brand car model, the probability distribution of x can be approximated by a normal distribution with a mean of 27 and standard deviation of 3.

a. If you were to buy this model of automobile, what is the probability that you would purchase one that averages less than 20 miles per gallon for in-city driving? In other words, find $P(x < 20)$.

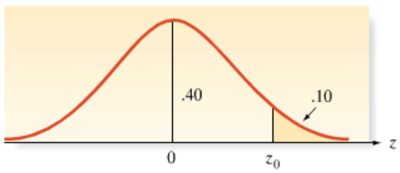
b. Suppose you purchase one of these new models and it does get less than 20 miles per gallon for in-city driving. Should you conclude that your probability model is incorrect?

Using the normal table to find z value:

Examples: Given a specific probability (or area), find z value corresponding to this probability.

Q: find the value of z, call it z_0 , such that $P(z \geq z_0) = 0.10$

(Note: Z table provides us two informations: z-value and the area (or probability) between 0 and z, so we know one information, we can know the other one.
Now we want to find z_0, we need find the area (or probability) between 0 and z_0 first, then we can check this probability in the entries of the table, then back to corresponding z value.)



Example1: find the value z_0 , such that $P(z \geq z_0) = 0.025$

Example 2: Find z_0 .

1. $P(z \geq z_0) = 0.10, \quad then\ z_0 =$
   $P(z < z_0) = 0.90, \quad then\ z_0 =$

2. $P(z \geq z_0) = 0.5, \quad then\ z_0 =$
   $P(z < z_0) = 0.5, \quad then\ z_0 =$

3. $P(z \geq z_0) = 0.25, \quad then\ z_0 =$
   $P(z < z_0) = 0.75, \quad then\ z_0 =$

4. $P(z \geq z_0) = 0.05, \quad then\ z_0 =$
   $P(z < z_0) = 0.95, \quad then\ z_0 =$

5. $P(z \geq z_0) = 0.025, \quad then\ z_0 =$
   $P(z < z_0) = 0.025, \quad then\ z_0 =$

6. $P(z < z_0) = 0.10$, then $z_0 =$

7. $P(z < z_0) = 0.01, \quad then\ z_0 =$

8. $P(z \leq z_0) = 0.80, \quad then\ z_0 =$

*Questions: (true or false)

1. If $P(z > z_0) < 0.50$, *then* $z_0 > 0$.
   If $P(z > z_0) > 0.50$, *then* $z_0 < 0$.

Occasionally we will be given a probability and we will want to find the values of x, the normal random variable that correspond to the probability.

Steps for finding a x-value correspond to a specific probability:

1. ________ the normal curve associated with the variable.
2. ________ the specific area of interest.
3. Use Z-table to find _______ corresponding to the probability.

4. Convert z value to x-value by using ______________ (or __________)

Examples:
The random variable x has a normal distribution with $\mu = 40$ and $\sigma^2 = 36$. Find a value of x, say, x_0, such that

1. $P(x > x_0) = 0.0228$

2. $P(x \le x_0) = 0.75$

3. $P(x \le x_0) = 0.10$

4. $P(x \ge x_0) = 0.50$

5. $P(x \le x_0) = 0.90$

6. $P(x \ge x_0) = 0.05$

Example#1: Suppose the scores, x, on a college entrance examination are normally distributed with a mean of 550 and a standard deviation of 100.
a. A certain prestigious university will consider for admission only those applicants whose scores exceed the 90th percentile of the distribution. Find the minimum score for admission consideration.

b. What is the 50th percentile examination score?

c. What is the 75th percentile examination score?

Example 2: A physical-fitness association is including the mile run in its secondary-school fitness test for students. The time for this event is approximately normally distributed with a mean of 450 seconds and a standard deviation of 40 seconds. If the association wants to designate the fastest 10% of secondary-school students as "Excellent", what time should the association set for this criterion?

Extra examples:

1. The board of examiners that administers the real estate broker's examination in a certain state found that the mean score on the test was 553 and the standard deviation was 72. If the board wants to set the passing score so that only the best 15% of all applicants pass, what is the passing score? Assume that the scores are normally distributed.

2. The board of examiners that administers the real estate broker's examination in a certain state found that the mean score on the test was 553 and the standard deviation was 72. If the board wants to set the passing score so that only the best 80% of all applicants pass, what is the passing score? Assume that the scores are normally distributed.

Learning objective of Chapter 5:

1. Understand normal distribution and standard normal distribution
2. Using z-table, find probability (area)
3. Using z-table, find z values
4. Find probability for a normal R.V.
5. Given probability, find a specific x value for a normal R.V.