1. Does the sequence \{1, 1/1, 2, 1/2, 3, 1/3, 4, 1/4, 5, 1/5, \ldots\} converge? Does it have any convergent subsequences?

**Answer:** The sequence does not converge. Let \(\{x_n\}\) denote the sequence. Suppose it converged to \(x\). Then for \(n\) large enough, \(|x_n - x| < 1\). Then \(|x_n - x_{n+1}| < |x_n - x| + |x - x_{n+1}| < 2\) for \(n\) large. But this doesn’t happen. For \(n > 3\), \(|x_n - x_{n+1}| > 2\). Therefore the sequence cannot converge.

The subsequence of even terms, \{1, 1/2, 1/3, \ldots\}, converges to zero.

2. Consider the dynamical system
   \[
   \mathbf{x}_{t+1} = \begin{bmatrix} 3/2 & 1/2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_t
   \]

   a) Find the eigenvalues of the matrix.

   **Answer:** The eigenvalue equation is \((3/2 - \lambda)(1 - \lambda) - 1/2 = 0\). Solving the quadratic yields \(\lambda = 2\) and \(\lambda = 1/2\).

   b) Find the corresponding eigenvectors.

   **Answer:** We start with \(\lambda = 2\) and refer to the matrix as \(A\).

   \[ (A - \lambda)\mathbf{v} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \]

   The vector \((1, 1)^T\) is a solution. Similarly, \((1, -2)^T\) solves the corresponding equation for \(\lambda = 1/2\).

   c) Are there any non-zero \(\mathbf{x}_0\) so that \(\mathbf{x}_t \to \mathbf{0}\)?

   **Answer:** Yes. Consider an eigenvector associated with \(\lambda = 1/2\). Set \(\mathbf{x}_0 = (1, -2)^T\). Then \(A^t\mathbf{x}_0 = (1/2)^t\mathbf{x}_0\). Since \((1/2)^t \to 0\), \(A^t\mathbf{x}_0 \to \mathbf{0}\).

3. Two embedding problems:

   a) Consider \(A = (0, 1) \subseteq \mathbb{R}\), and \(B = \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1\} \subseteq \mathbb{R}^2\). Is \(A\) open? Is \(B\) open? If the two differ, explain why embedding \(A\) in \(\mathbb{R}^2\) changed the result.

   **Answer:** The set \(A\) is open. Let \(x \in A\) and set \(\epsilon = \min(x, 1 - x)\). Then \(B_\epsilon(x) \subseteq A\). Since \(A\) contains an open ball about each of its points, it is an open set.

   The set \(B\) is not open. Let \((x, 0) \in B\) and consider \(B_\epsilon(x, 0)\). The point \((x, \epsilon/2) \in B_\epsilon(x, 0)\), but is not in \(B\). Thus there is a point in \(B\) so that no open ball about that point is contained in \(B\). (In fact, this is true of all points in \(B\)). Therefore \(B\) is not open.

   The difference occurs because of the extra dimension. Balls in \(\mathbb{R}^2\) contain points not on the horizontal axis, and so cannot be contained in sets restricted to the horizontal axis. With balls in \(\mathbb{R}^1\), this is not an issue.

   b) Now suppose \(C\) is a closed set in \(\mathbb{R}^2\). Show that \(D = \{(x, y, 0) : (x, y) \in C\}\) is closed in \(\mathbb{R}^3\).

   **Answer:**

   Let \((x_n, y_n, z_n) \in D\) with \((x_n, y_n, z_n) \to (x, y, z)\). Since \(z_n = 0\), \(z = 0\). Because \(C\) is closed and \((x_n, y_n) \in C\) and \((x_n, y_n) \to (x, y)\), we find \((x, y) \in C\), so \((x, y, z) = (x, y, 0) \in D\). Thus \(D\) is closed because it contains all of its limit points.

4. Let \(f(x, y, z) = x^2 + 3y + z^3 - 5\).

   a) Find an \((x_0, y_0, z_0)\) satisfying \(f(x_0, y_0, z_0) = 0\).
**Answer:** The point \((1,1,1)\) works.

**b)** Can \(x\) be expressed as a function \(g(y,z)\) in some neighborhood of \((x_0,y_0,z_0)\)?

**Answer:** Since \(\frac{\partial f}{\partial x} = 2x\), \(\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} = 2\). The Implicit Function Theorem yields such a function \(g\). Alternatively, note that \(g(y,z) = (5 - 3y - z^3)^{1/2}\) works.

**c)** Compute \(dg\).

**Answer:** By the Implicit Function Theorem, \(dg = -(1/2)x(\partial f/\partial y, \partial f/\partial z) = -(3/2)(3, 3z^2)\). At \((1,1,1)\), this has the value \((-3/2, -3/2)\).

5. Consider the function \(u(x,y) = x + \sqrt{y}\).

**a)** Does \(u\) attain a maximum on the set \(A = \{(x,y) : x, y \geq 0, px + y \leq 3\}\)? Why?

**Answer:** The function \(u\) is continuous. The set \(A\) is a budget set in \(\mathbb{R}^2\) with strictly positive prices. We saw in class that such a set is compact. The Weierstrass Theorem says that any continuous function attains a maximum on a compact set.

**b)** Does \(u\) attain a maximum on the set \(B = \{(x,y) : y \geq 0, px + y \leq 3\}\)? Why?

**Answer:** This is a trickier problem. As was the case with the cost minimization problem done in class, we must simplify the problem before solving it. First, note that \(u\) is increasing in both \(x\) and \(y\). It follows that any maximum must occur along the constraint \(px + y = 3\). So \(x = (3 - y)/p\). Substituting into \(u\), we reduce our problem to maximizing \((3 - y)/p + \sqrt{y}\) over the set \(y \geq 0\). Note that the maximum must be at least \(3/p\), which is the value taken at \(y = 0\). It is easy to see that \(\lim_{y \to \infty}(3 - y)/p + \sqrt{y} = -\infty\), so there is some \(b\) with \((3 - y)/p + \sqrt{y} < 3/p\) for \(y > b\). We can now remove any \(y > b\) from the set we are maximizing over, and focus on whether \((3 - y)/p + \sqrt{y}\) can be maximized over \([0,b]\). But now we are maximizing a continuous function over a compact set, and Weierstrass’s Theorem tells us that there is a maximum. This is also a maximum for the original problem.