Mathematical Economics Midterm #2, November 6, 2000

1. Does the sequence $\{1, 1/1, 2, 1/2, 3, 1/3, 4, 1/4, 5, 1/5, \dots\}$ converge? Does it have any convergent subsequences?

   Answer: The sequence does not converge. Let $\{x_n\}$ denote the sequence. Suppose it converged to x. Then for n large enough, $|x_n - x| < 1$. Then $|x_n - x_{n+1}| < |x_n - x| + |x - x_{n+1}| < 2$ for n large. But this doesn't happen. For $n > 3$, $|x_n - x_{n+1}| > 2$. Therefore the sequence cannot converge.

   The subsequence of even terms, $\{1/1, 1/2, 1/3, \dots\}$, converges to zero.

2. Consider the dynamical system
$$\mathbf{x}_{t+1} = \begin{bmatrix} 3/2 & 1/2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_t$$

   a) Find the eigenvalues of the matrix.

   Answer: The eigenvalue equation is $(3/2 - \lambda)(1 - \lambda) - 1/2 = 0$. Solving the quadratic yields $\lambda = 2$ and $\lambda = 1/2$.

   b) Find the corresponding eigenvectors.

   Answer: We start with $\lambda = 2$ and refer to the matrix as A.
$$(A - \lambda)\mathbf{v} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [\,0\,].$$

   The vector $(1, 1)^T$ is a solution. Similarly, $(1, -2)^T$ solves the corresponding equation for $\lambda = 1/2$.

   c) Are there any non-zero $\mathbf{x}_0$ so that $\mathbf{x}_t \to \mathbf{0}$?

   Answer: Yes. Consider an eigenvector associated with $\lambda = 1/2$. Set $\mathbf{x}_0 = (1, -2)^T$. Then $A^t\mathbf{x}_0 = (1/2)^t\mathbf{x}_0$. Since $(1/2)^t \to 0$, $A^t\mathbf{x}_0 \to \mathbf{0}$.

3. Two embedding problems:

   a) Consider $A = (0, 1) \subset \mathbb{R}$, and $B = \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1\} \subset \mathbb{R}^2$. Is A open? Is B open? If the two differ, explain why embedding A in $\mathbb{R}^2$ changed the result.

   Answer: The set A is open. Let $x \in A$ and set $\epsilon = \min(x, 1 - x)$. Then $B_\epsilon(x) \subset A$. Since A contains an open ball about each of its points, it is an open set.

   The set B is not open. Let $(x, 0) \in B$ and consider $B_\epsilon(x, 0)$. The point $(x, \epsilon/2) \in B_\epsilon(x, 0)$, but is not in B. Thus there is a point in B so that no open ball about that point is contained in B. (In fact, this is true of all points in B). Therefore B is not open.

   The difference occurs because of the extra dimension. Balls in $\mathbb{R}^2$ contain points not on the horizontal axis, and so cannot be contained in sets restricted to the horizontal axis. With balls in $\mathbb{R}^1$, this is not an issue.

   b) Now suppose C is a closed set in $\mathbb{R}^2$. Show that $D = \{(x, y, 0) : (x, y) \in C\}$ is closed in $\mathbb{R}^3$.

   Answer:

   Let $(x_n, y_n, z_n) \in D$ with $(x_n, y_n, z_n) \to (x, y, z)$. Since $z_n = 0$, $z = 0$. Because C is closed and $(x_n, y_n) \in C$ and $(x_n, y_n) \to (x, y)$, we find $(x, y) \in C$, so $(x, y, z) = (x, y, 0) \in D$. Thus D is closed because it contains all of its limit points.

4. Let $f(x, y, z) = x^2 + 3y + z^3 - 5$.

   a) Find an (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

Answer: The point $(1,1,1)$ works.

b) Can x be expressed as a function $g(y,z)$ in some neighborhood of (x_0,y_0,z_0)?

Answer: Since $\partial f/\partial x = 2x$, $\partial f/\partial x = 2$ at (x_0,y_0,z_0). The Implicit Function Theorem yields such a function g. Alternatively, note that $g(y,z) = (5-3y-z^3)^{1/2}$ works.

c) Compute dg.

Answer: By the Implicit Function Theorem, $dg = (-1/2x)(\partial f/\partial y, \partial f/\partial z) = -(3/2)(1,3z^2)$. At $(1,1,1)$, this has the value $(-3/2,-3/2)$.

5. Consider the function $u(x,y) = x + \sqrt{y}$.

a) Does u attain a maximum on the set $A = \{(x,y) : x,y \geq 0,\ px+y \leq 3\}$? Why?

Answer: The function u is continuous. The set A is a budget set in $\mathbb{R}^2$ with strictly positive prices. We saw in class that such a set is compact. The Weierstrass Theorem says that any continuous function attains a maximum on a compact set.

b) Does u attain a maximum on the set $B = \{(x,y) : y \geq 0,\ px+y \leq 3\}$? Why?

Answer: This is a trickier problem. As was the case with the cost minimization problem done in class, we must simplify the problem before solving it. First, note that u is increasing in both x and y. It follows that any maximum must occur along the constraint $px+y=3$. So $x = (3-y)/p$. Substituting into u, we reduce our problem to maximizing $(3-y)/p + \sqrt{y}$ over the set $y \geq 0$. Note that the maximum must be at least $3/p$, which is the value taken at $y=0$. It is easy to see that $\lim_{y\to\infty}(3-y)/p + \sqrt{y} = -\infty$, so there is some b with $(3-y)/p + \sqrt{y} < 3/p$ for $y > b$. We can now remove any $y > b$ from the set we are maximizing over, and focus on whether $(3-y)/p + \sqrt{y}$ can be maximized over $[0,b]$. But now we are maximizing a continuous function over a compact set, and Weierstrass's Theorem tells us that there is a maximum. This is also a maximum for the original problem.