Mathematical Economics Midterm #1, October 4, 2001

1. (Eigenvalues and eigenvectors) Let $A = \begin{pmatrix} \frac{9}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{6}{5} \end{pmatrix}$.

   a) Find the eigenvalues of A.

   The eigenvalues are found by solving $\det(A - \lambda I) = 0$ for λ. Now $\det(A - \lambda I) = (\frac{9}{5} - \lambda)(\frac{6}{5} - \lambda) - \frac{4}{25} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$. It follows that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$.

   b) Find eigenvectors corresponding to the eigenvalues.

   For $\lambda_1 = 2$, consider $(A - 2I)\mathbf{v} = \mathbf{0}$. Thus

$$\begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}v_1 + \frac{2}{5}v_2 \\ \frac{2}{5}v_1 - \frac{4}{5}v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   This implies $v_1 = 2v_2$. Setting $v_2 = 1$ we find the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   For $\lambda_1 = 1$, consider $(A - I)\mathbf{v} = \mathbf{0}$. Thus

$$\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5}v_1 + \frac{2}{5}v_2 \\ \frac{2}{5}v_1 + \frac{1}{5}v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   This implies $2v_1 = -v_2$. Setting $v_1 = 1$ we find the eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

   c) Are the eigenvectors in part (b) orthonormal? If not, can you find a set of orthonormal eigenvectors? Demonstrate.

   Although $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ (the vectors are orthogonal), $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{5}$. Thus the vectors are not orthonormal.

   Since scalar multiples of eigenvectors are also eigenvectors, we can fix this by dividing the eigenvectors by their norms. This yields $\mathbf{w}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{w}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2. (Determinants) Find the determinant of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}.$$

Hint: It's easy, if you use a trick from the homework.

The trick is to put the matrix into row-echelon form. Provided we have not interchanged rows, the determinant is then the product of the diagonal entries.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 15 \\ 0 & 7 & 26 & 63 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 12 & 42 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

It follows that the determinant is $1 \times 1 \times 2 \times 6 = 12$.

3. (Linear Systems) Consider the linear system

$$\begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6. \end{aligned}$$

Determine whether this system has solutions. How many solutions are there?

This system has augmented matrix

$$\begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 4 & -1 & 1 & 3 \\ 3 & 7 & 1 & 1 & 6 \end{pmatrix}.$$

We row-reduce the augmented matrix to determine whether there are solutions:

$$\begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 4 & -1 & 1 & 3 \\ 3 & 7 & 1 & 1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 0 & 5 & -4 & 2 & 3 \\ 0 & 10 & -8 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 0 & 5 & -4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This can be further row-reduced to

$$\begin{pmatrix} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since both the augmented matrix has the same rank (2) as the unaugmented matrix, the system has a solution. Here both y and z are free variables, and so the system has many solutions.

4. (Linear Systems) Demand for good 1 is $d_1 - ap_1 + \frac{1}{2}bp_2$; demand for good 2 is $d_2 + \frac{1}{2}ap_1 - bp_2$; the supply of good i is s_i. Here a, b, d_i, and s_i are all positive, and $s_i > d_i$.

   a) What system of equations do you get when you set supply equal to demand in both markets?

$$\begin{aligned} d_1 - ap_1 + \frac{1}{2}bp_2 &= s_1 \\ d_2 + \frac{1}{2}ap_1 - bp_2 &= s_2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} -ap_1 + \frac{1}{2}bp_2 &= s_1 - d_1 \\ \frac{1}{2}ap_1 - bp_2 &= s_2 - d_2 \end{aligned}$$

   b) What criterion must be met in order to solve for p_1 and p_2? Is it satisfied?

   The determinant

$$\begin{vmatrix} -a & \frac{1}{2}b \\ \frac{1}{2}a & -b \end{vmatrix} = ab - \frac{1}{4}ab = \frac{3}{4}ab \neq 0.$$

   This is satisfied since a and b are both positive.

   c) What additional conditions must be satisfied in order to get positive equilibrium prices p_i?

   Equilibrium prices are always negative. Using Cramer's rule, we find

$$p_1 = -\frac{4}{3a}[s_1 - d_1 + \frac{1}{2}(s_2 - d_2)]$$

   and

$$p_2 = -\frac{4}{3b}[\frac{1}{2}(s_1 - d_1) + s_2 - d_2].$$

   Given the conditions on the parameters, prices are always negative.

5. (Norms) Let $\|\cdot\|$ be the Euclidean norm. Show that $\mathbf{x}\cdot\mathbf{y} = \frac{1}{4}\{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\}$.

   We expand the right-hand side:

$$\frac{1}{4}\{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\} = \frac{1}{4}\left\{\|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} - \|\mathbf{y}\|^2\right\} = \mathbf{x}\cdot\mathbf{y}$$

   This result is called the *polarization identity*.