Mathematical Economics Midterm #2, November 8, 2001

1. Consider the sequence $x_n = 2^n + (-2)^n + 1/n^2$.

   a) Does the sequence converge? If so, what is its limit?

   Answer: The sequence does not converge. In fact, $|x_n - x_{n+1}| > 2^n$, so the terms get farther apart.

   b) If the sequence doesn't converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   Answer: Yes, it has convergent subsequences. One such subsequence is the subsequence of odd terms, $x_{n_j} = x_{2j+1} = 1/(2j+1)^2$. This subsequence converges to 0.

2. Find all local and global maxima and minima of the function $f(x, y) = (2/3)x^3 + x^2 + 2y^2 - 2xy + 4x - 10y$.

   Answer: First compute $df = (2x^2 + 2x - 2y + 4, 4y - 2x - 10)$. Setting $df = (0, 0)$, we find $2y = x + 5$ from the second equation. Substituting in the first equation, this implies $2x^2 + x - 1 = 0$. This has solutions $x = -1$ and $x = 1/2$. It follows that the critical points are $(-1, 2)$ and $(1/2, 11/4)$.

   We next compute the Hessian:

$$d^2 f = \begin{bmatrix} 4x + 2 & -2 \\ -2 & 4 \end{bmatrix}.$$

   At $(-1, 2)$, we obtain $H_1 = -2 < 0$ and $H_2 = -8 - 4 = -12 < 0$. The Hessian is indefinite and $(-1, 2)$ is neither a local minimum nor local maximum. At $(1/2, 11/4)$, we obtain $H_1 = 4 > 0$ and $H_2 = 16 - 4 = 12 > 0$. The Hessian is now positive definite and $(1/2, 11/4)$ is a local minimum.

   There is no global maximum. In fact $\lim_{x\to+\infty} f(x, 0) = +\infty$. There is also no global minimum since $\lim_{x\to-\infty} f(x, 0) = -\infty$

3. Consider the quadratic form $Q(x, y) = x^2 + 8xy + y^2$.

   a) Does the quadratic form have a maximum or minimum?

   Answer: This quadratic form has associated symmetric matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}.$$

The first leading principal minor is $1 > 0$, while the second leading principal minor is $1-16 = -15 < 0$. This violates the sign patterns for positive or negative definite, so the matrix is indefinite. The form has neither maximum nor minimum.

b) Now impose the constraint $2x+3y = 0$. Does the quadratic form have a constrained maximum or minimum?

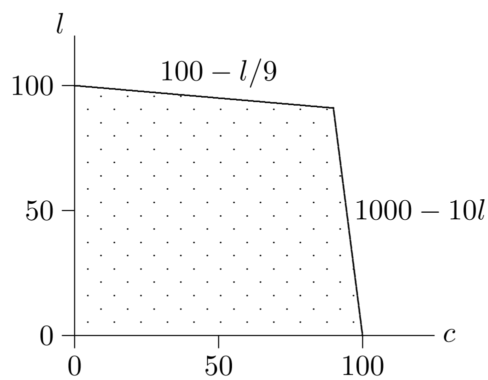
Answer: In this case we consider the bordered matrix:

$$H = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}.$$

The only princial minor to consider is the determinant of the whole matrix, which is $35 > 0$. Since $n = 2$ and so $(-1)^n = 1$, we have $(-1)^n \det H = 1$. This implies the form attains a maximum at $(0, 0)$.

4. Suppose that utility is a continuous function from $\mathbb{R}^2_+$ to $\mathbb{R}$. The budget set is $\{(c,l) : (c,l) \geq \mathbf{0}, c \leq 1000 - 10l, c \leq 100 - l/9\}$.

a) Draw the budget set. **Answer:**



Budget Set

b) Does the consumer's problem have a solution? That is, can utility be maximized over the budget set? Explain.

Answer: Yes, the consumer's problem has a solution. It suffices to show the budget set is compact since the Weierstrass Theorem will imply there is a solution.

The budget set is the interesection of four sets: $\{(c,l) : l \geq 0\}$, $\{(c,l) : c \geq 0\}$, $\{(c,l) : c \leq 1000-10l\}$, and $\{(c,l) : c \leq 100-l/9\}$. Each of these sets is the inverse image of a closed interval under a continuous function, and so each is closed. As the intersection of closed sets, the budget set is closed.

Since $l \geq 0$, $c \leq 100 - l/9 \leq 100$. Since $c \geq 0$, $10l \leq 1000 - c \leq 1000$, so $l \leq 100$. The budget set is contained in $[0, 100] \times [0, 100]$ (see the diagram), and thus is bounded. Since it is both closed and bounded, it is compact. By the Weierstrass Theorem, the consumer's problem has a solution.

Note: Polygonal budget sets of this type arise when consumers face progressive taxation, where the number of sides depends on the number of tax brackets. The phase-out of welfare benefits as income increases has a similar effect.

5. Let $f(x, y, z) = x^2 + xy + y^2 + z^3 - 4$.

a) Find a point (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

Answer: There are many solutions: The point $(x_0, y_0, z_0) = (1, 1, 1)$ seemed to be the consensus choice, but $(0, 2, 0)$, $(2, 0, 0)$, $(5, 1, -3)$, $(1, 5, -3)$, $(-2, 1, 1)$, $(1, -2, 1)$, and $(2, 2, 2)$ are other integral solutions. Of course, you are not restricted to integers, but they often make calculation easier.

b) Can x be expressed as a function $g(y, z)$ in some neighborhood of (x_0, y_0, z_0)?

Answer: We use the point $(x_0, y_0, z_0) = (1, 1, 1)$. Here $\partial f/\partial x = 2x_0 + y_0 = 3 \neq 0$. The Implicit Function Theorem now implies that we can write $x = g(y, z)$ in some neighborhood of $(1, 1, 1)$.

Note: In this case, we can use the quadratic formula to find an expression for g,

$$g(y, z) = \frac{-y + \sqrt{16 - 3y^2 - 4z^3}}{2}.$$

Of course, this is only valid for $16 - 3y^2 - 4z^3 \geq 0$. Notice that the positive square root is chosen so that $g(y_0, z_0) = g(1, 1) = 1 = x_0$. Had we used $(-2, 1, 1)$ as our reference point, we would use the negative square root.

c) Compute $dg(y_0, z_0)$.

Answer: By the Implicit Function Theorem,

$$\begin{aligned} dg(y_0, z_0) &= -\left(\frac{\partial f}{\partial x}\right)^{-1} \begin{bmatrix} \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} \\ &= \frac{-1}{2x_0 + y_0} \begin{bmatrix} x_0 + 2y_0 \\ 3z_0 \end{bmatrix} \\ &= -\frac{1}{3} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \end{aligned}$$

Notice that this is an easier computation than using the formula for g given in part (*b*).