Mathematical Economics Final, December 13, 2001

1. Consider the function $f(x, y) = x^3 + xy + y^2 - x/2 + y$. Find and classify (maximum, minimum, saddlepoint) all critical points of f.

   Answer: The derivative is $df = (3x^2 + y - 1/2, x + 2y + 1)$. Setting this to zero, we find $3x^2 + y - 1/2 = 0$ and $x + 2y = -1$. The second equation can be rewritten $y = -(1 + x)/2$. Substituting in the first equation yields $3x^2 - x/2 - 1 = 0$. This has solutions $x = 2/3$ and $x = -1/2$. The critical points are then $(2/3, -5/6)$ and $(-1/2, -1/4)$.

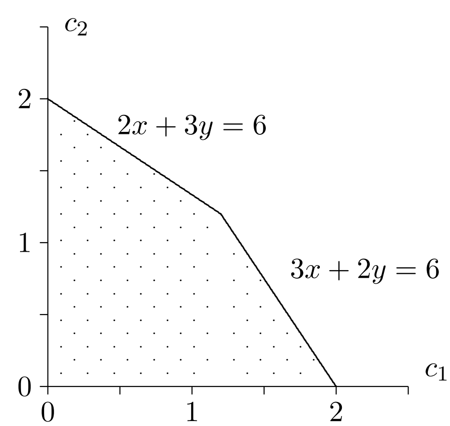
   The Hessian is:
$$H = d^2 f = \begin{bmatrix} 6x & 1 \\ 1 & 2 \end{bmatrix}.$$

   Thus $H_1 = 6x$ and $H_2 = 12x - 1$. When $x = 2/3$, $H_1 = 4 > 0$ and $H_2 = 7 > 0$. The Hessian is positive definite, and so $(2/3, -5/6)$ is a local minimum. (There is no global maximum or minimum since $f(x, 0) = x^3 - x/2$ is unbounded above and below.) When $x = -1/2$, $H_1 = -3 < 0$ and $H_2 = -2 < 0$. The Hessian is indefinite, and so $(-1/2, -1/4)$ is a saddlepoint.

2. A consumer has utility function $u(x, y) = \ln x + \ln y$. The consumer consumes non-negative quantities of both goods, subject to two budget constraints: $3x + 2y \le 6$ and $2x + 3y \le 6$. Find (x^*, y^*) that maximizes utility subject to the above four constraints. Be sure to check the constraint qualification and second-order conditions.

   Answer: This sort of problem can arise when one of the goods is rationed via rationed coupons, and there is a market for ration coupons where the relative price for coupons is different than for goods. We first consider constraint qualification. There are four constraints: $3x + 2y - 6 \le 0$, $2x + 3y - 6 \le 0$, $-x \le 0$, and $-y \le 0$. The matrix of derivatives of the constraints is:
$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Constraint qualification will be satisfied as long as at most 2 constraints bind. Examining the constraints shows that at most 2 can bind (see the diagram).



Budget Set

The Lagrangian is $L = \ln x + \ln y - \lambda(3x + 2y - 6) - \mu(2x + 3y - 6) + \nu_x x + \nu_y y$. The first-order conditions are $1/x - 3\lambda - 2\mu + \nu_y = 0$ and $1/y - 2\lambda - 3\mu + \nu_y = 0$. These equations cannot be satisfied if $x = 0$ or $y = 0$, so $x > 0$ and $y > 0$. Complementary slackness now implies $\nu_x = \nu_y = 0$. The first-order equations become $1/x = 3\lambda + 2\mu$ and $1/y = 2\lambda + 3\mu$.

There are now 3 cases.

Case 1: If both remaining constraints bind, $x = y = 6/5$. It follows that $\lambda = \mu = 1/6$.

Case 2: If $2x + 3y < 6$, $\mu = 0$ by complementary slackness. This implies $y = 1/2\lambda$ and $x = 1/3\lambda$. Because $\lambda > 0$, $3x + 2y = 6$. Substituting, we find $\lambda = 1/3$, so $x = 1$ and $y = 1.5$. But now $2x + 3y = 2 + 4.5 = 6.5$, contradicting $2x + 3y < 6$.

Case 3: If $3x + 2y < 6$, $\lambda = 0$ by complementary slackness. This implies $y = 1/3\mu$ and $x = 1/2\mu$. Because $\mu > 0$, $2x + 3y = 6$ by complementary slackness. But now $\mu = 1/3$, so $x = 1.5$ and $y = 1$. This implies $3x + 2y = 6.5$, contradicting $2x + 3y < 6$.

Thus $(x, y) = (6/5, 6/5)$ is our only remaining critical point. The Hessian is:

$$d^2L = \begin{bmatrix} -x^{-2} & 0 \\ 0 & -y^{-2} \end{bmatrix},$$

which is negative definite. Thus we maximized utility. (We don't need to check the bordered Hessian when the Hessian itself is negative definite).

3. Find the rank and all eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix}.$$

a) What is the rank of A?

Answer: We compute $\det A$ by expanding the determinant along the first row. It is 4 times the determinant of the 3×3 sub-matrix in the lower right. But this sub-matrix has determinant 1, so $\det A = 4$. Thus A is invertible and has rank 4.

b) Find all eigenvalues of A.

Answer: We set $0 = \det(A - \lambda I) = (4 - \lambda)(1 - \lambda)^3$. This has solutions $\lambda = 4$ and $\lambda = 1$, which are the eigenvalues of A.

4. Minimize $p_x x + p_y y$ subject to the constraint $x^2y^4 \geq 432$. Be sure to check constraint qualification and the second-order conditions.

Answer: We first differentiate the constraint g, obtaining $dg = (2xy^4, 4x^2y^3)$. As $x^2y^4 \geq 432$, neither x nor y is zero, and dg has rank 1. Constraint qualification is satisfied.

The Lagrangian is $L = p_x x + p_y y - \lambda(x^2y^4 - 432)$. The first-order conditions are $p_x - 2\lambda xy^4 = 0$ and $p_y - 4\lambda x^2y^3 = 0$. Presuming $p_x > 0$ or $p_y > 0$, we find that $\lambda > 0$, so $x^2y^4 = 432$. Dividing one first-order condition by the other, we obtain $y = 2p_x x/p_y$. Substituting into the constraint yields $16p_x^4x^6/p_y^4 = 432$, or $x^6 = 27p_y^4/p_x^4$. Thus $x = 3^{1/2}p_y^{2/3}p_x^{-2/3}$ and $y = (2\sqrt{3})p_x^{1/3}p_y^{-1/3}$. Moreover, $\lambda = 3^{1/2}p_x^{1/3}p_y^{2/3}/864$.

The bordered Hessian of L is:

$$\begin{bmatrix} 0 & 2xy^4 & 4x^2y^3 \\ 2xy^4 & -2\lambda y^4 & -8\lambda xy^3 \\ 4x^2y^3 & -8\lambda xy^3 & -12\lambda x^2y^2 \end{bmatrix}.$$

Since there is one effective constraint, we must check that the last leading principal minor has sign $(-1)^1$ (is negative). The determinant of the bordered Hessian is $-64\lambda x^4y^{10} - 64\lambda x^4y^{10} + 32\lambda x^4y^{10} + 48\lambda x^4y^{10} = -48\lambda x^4y^{10} < 0$, implying that we have a minimum.

5. A consumer consumes two consumption goods $\mathbf{c} = (c_1, c_2) \geq (0,0)$. Utility is a continuous function of $\mathbf{c}$. The consumer's endowment is $(0,10)$. Some of good two is sold at price 10 in order to finance consumption of good one, which has price $p > 0$. Income from the sale of good two is $I = 10(10 - c_2)$. Income is taxed according to the following tax schedule:

$$\tau(I) = \begin{cases} .1I & \text{if } 0 \leq I \leq 20 \\ 2 + .2(I-20) & \text{if } 20 \leq I \leq 50 \\ 5 + .5(I-50) & \text{if } 50 \leq I \leq 80 \\ 20 + .9(I-80) & \text{if } 80 \leq I \leq 100 \end{cases}$$

a) The consumer must pay taxes and pay for good one out of his income. Use this fact to write the budget constraint for the consumer in terms of (c_1, c_2), p and the tax function τ.

Answer: Income is $I = 10(10 - c_2) = 100 - 10c_2$. Income is used to buy good one and to pay taxes. Thus $pc_1 + \tau(100 - 10c_2) \leq 100 - 10c_2$, or $pc_1 + 10c_2 + \tau(100 - 10c_2) \leq 100$.

b) Is the budget set compact? Explain.

Answer: Yes, the budget set is compact (closed and bounded).

First we show the budget set is bounded. Note $pc_1 + 10c_2 \leq pc_1 + 10c_2 + \tau(100 - 10c_2) \leq 100$. It follows that the budget set is contained in the usual budget set with prices $(p, 10) \gg 0$ and income 100. As the usual budget set is bounded, so is our budget set.

Second, we show the budget set is closed. Let (c_1^n, c_2^n) be in the budget set and have limit (c_1, c_2). Since $c_1^n \geq 0$ and $c_2^n \geq 0$, we can take the limit to find $c_1 \geq 0$ and $c_2 \geq 0$. Also, $pc_1^n + 10c_2^n + \tau(100 - 10c_2^n) \leq 100$. Provided τ is continuous, we can take the limit to find $pc_1 + 10c_2 + \tau(100 - 10c_2) \leq 100$, which implies the budget set is closed.

Since τ is a piecewise linear function, we need only show that the pieces fit together, that the left and right limits are equal. The limites are equal since

$$\begin{aligned} \lim_{I \to 20^-} .1I = 2 &= \lim_{I \to 20^+} [2 + .2(I-20)]; \\ \lim_{I \to 50^-} [2 + .2(I-20)] = 5 &= \lim_{I \to 50^+} [5 + .5(I-50)]; \\ \lim_{I \to 80^-} [5 + .5(I-50)] = 20 &= \lim_{I \to 80^+} [20 + .9(I-80)]. \end{aligned}$$

c) Does the consumer's utility maximization problem have a solution when the budget constraint from (a) is used?

Answer: Since the budget set is compact and the utility function is continuous, we appeal to the Weierstrass Theorem and conclude that utility has a maximum over the budget set.