Mathematical Economics Midterm #2, November 12, 2002

1. Consider the sequence $x_n = (-1)^n + n/(n^2+1)$.

   a) Does it converge? If so, what is its limit?

   b) If it doesn't converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   Answer: The sequence does not converge. One way to see that is to consider $\lim_n |x_n - x_{n+1}| = 2$. Since the terms do not get closer together (i.e., it is not Cauchy), the sequence doesn't converge.

   However, it is easy to find convergent subsequences. The even terms converge to 1 and the odd terms converge to -1.

2. Let $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$.

   a) Find the eigenvalues of A.

   b) Find eigenvectors corresponding to the eigenvalues.

   c) Are the eigenvectors in part (b) orthonormal? If not, can you find a set of orthonormal eigenvectors? Demonstrate.

   Answer:

   a) The eigenvalue equation is $0 = (9-\lambda)(6-\lambda) - 4 = 50 - 15\lambda + \lambda^2$. It has solutions $\lambda = 5$ and $\lambda = 10$.

   b) To find an eigenvector corresponding to $\lambda = 10$, we solve

   $$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   One solution is $\mathbf{v} = (2, -1)^T$. To find an eigenvector corresponding to $\lambda = 5$, we solve

   $$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   One solution is $\mathbf{u} = (1, 2)^T$.

   c) Although $\mathbf{u} \cdot \mathbf{v} = 0$, which shows the eigenvectors are orthogonal, they do not have unit norm. We divide by their norms to get orthonormal eigenvectors:

   $$\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}.$$

3. Robinson Crusoe has utility function $u(x, y) = x^2 + y^2$. Crusoe has a production possibilities set given by $\{(x, y) : x \geq 0, y \geq 0, x + y^2 \leq 4\}$.

   a) Is the production possibility set a closed set? Is it a bounded set?

   b) Show that Crusoe's problem of maximizing utility over his production set has a solution.

   Answer:

   a) The set is closed. There are many ways to see this, one is to realize that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = x + y^2$ are all continuous. Then $f^{-1}([0, \infty))$, $g^{-1}([0, \infty))$ and $h^{-1}((-\infty, 4])$ are all closed sets (as the inverse image of closed intervals). The production possibility set is closed because it is the intersection of 3 closed sets.

   The production set is bounded because $y^2 \leq 4$ and $y \geq 0$ imply $0 \leq y \leq 2$ and $0 \leq x \leq 4$.

   b) Since the utility is continuous (we know all polynomials are continuous) and the production set is compact (closed and bounded), the Weierstrass Theorem applies to yield a maximum.

4. Suppose a firm's production function is $Q = K^{1/3}L^{2/3}$ and that $K = 1000$ and $L = 125$.

   a) How much can the firm produce?

   b) What are the marginal products of capital (K) and labor (L)?

   c) Suppose that the available capital falls by 2 units, while labor increases by 5 units. Without plugging the new numbers for K and L into the production function, compute approximately how much the firm can now produce.

   Answer:

   a) Maximum production is $Q = (1000)^{1/3}(125)^{2/3} = 250$.

   b) Now $MP_K = \partial Q/\partial K = \frac{1}{3}K^{-2/3}L^{2/3}$ and $MP_L = \partial Q/\partial L = \frac{2}{3}K^{1/3}L^{-1/3}$. Using $K = 1000$ and $L = 125$ yields $MP_K = \frac{1}{12}$ and $MP_L = \frac{4}{3}$.

   c) The change in production is $MP_K \Delta K + MP_L \Delta L = -1/6 + 20/3 = 6.5$. The resulting output level is 256.5.

5. Let $f(x, y, z) = x^2 + 3xy + 4y^2 + e^z - 9$.

   a) Find a point (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

   b) Can x be expressed as a function $g(y, z)$ in some neighborhood of (x_0, y_0, z_0)?

   c) Compute $dg(y_0, z_0)$.

Answer:

a) The point $(x_0, y_0, z_0) = (1, 1, 0)$ satisfies the equation.

b) We compute $\partial f/\partial x = 2x + 3y$. Plugging in $x = 1$ and $y = 1$, we obtain 5. The derivative is invertible (not zero). The Implicit Function Theorem then yields the desired function g. In fact, it is possible to compute $g(y, z) = [-3y + \sqrt{36 - 7y^2 - 4e^z}]/2$.

c) The derivative is given by

$$\begin{aligned} dg(1,1,0) &= -\frac{1}{5}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_{(1,1,0)} \\ &= -\frac{1}{5}\left(3x + 8y, e^z\right)\Big|_{(1,1,0)} \\ &= \left(-\frac{11}{5}, -\frac{1}{5}\right). \end{aligned}$$