1. Consider the sequence \( x_n = (-1)^n + n/(n^2 + 1) \).
   
   a) Does it converge? If so, what is its limit?

   b) If it doesn’t converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   **Answer:** The sequence does not converge. One way to see that is to consider \( \lim_{n \to \infty} |x_n - x_{n+1}| = 2 \). Since the terms do not get closer together (i.e., it is not Cauchy), the sequence doesn’t converge.

   However, it is easy to find convergent subsequences. The even terms converge to 1 and the odd terms converge to \(-1\).

2. Let \( A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \).
   
   a) Find the eigenvalues of \( A \).

   b) Find eigenvectors corresponding to the eigenvalues.

   c) Are the eigenvectors in part (b) orthonormal? If not, can you find a set of orthonormal eigenvectors? Demonstrate.

   **Answer:**

   a) The eigenvalue equation is \( 0 = (9 - \lambda)(6 - \lambda) - 4 = 50 - 15\lambda + \lambda^2 \). It has solutions \( \lambda = 5 \) and \( \lambda = 10 \).

   b) To find an eigenvector corresponding to \( \lambda = 10 \), we solve

   \[
   \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
   \]

   One solution is \( v = (2, -1)^T \). To find an eigenvector corresponding to \( \lambda = 5 \), we solve

   \[
   \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
   \]

   One solution is \( u = (1, 2)^T \).

   c) Although \( u \cdot v = 0 \), which shows the eigenvectors are orthogonal, they do not have unit norm. We divide by their norms to get orthonormal eigenvectors:

   \[
   \left( \frac{2}{\sqrt{5}} \right), \left( \frac{1}{\sqrt{5}} \right).
   \]
3. Robinson Crusoe has utility function \( u(x, y) = x^2 + y^2 \). Crusoe has a production possibilities set given by \( \{(x, y) : x \geq 0, y \geq 0, x + y^2 \leq 4\} \).

   a) Is the production possibility set a closed set? Is it a bounded set?

   b) Show that Crusoe’s problem of maximizing utility over his production set has a solution.

**Answer:**

   a) The set is closed. There are many ways to see this, one is to realize that the functions \( f(x, y) = x \), \( g(x, y) = y \), and \( h(x, y) = x + y^2 \) are all continuous. Then \( f^{-1}([0, \infty)) \), \( g^{-1}([0, \infty)) \) and \( h^{-1}((-\infty, 4]) \) are all closed sets (as the inverse image of closed intervals). The production possibility set is closed because it is the intersection of 3 closed sets.

   The production set is bounded because \( y^2 \leq 4 \) and \( y \geq 0 \) imply \( 0 \leq y \leq 2 \) and \( 0 \leq x \leq 4 \).

   b) Since the utility is continuous (we know all polynomials are continuous) and the production set is compact (closed and bounded), the Weierstrass Theorem applies to yield a maximum.

4. Suppose a firm’s production function is \( Q = K^{1/3}L^{2/3} \) and that \( K = 1000 \) and \( L = 125 \).

   a) How much can the firm produce?

   b) What are the marginal products of capital (\( K \)) and labor (\( L \))?

   c) Suppose that the available capital falls by 2 units, while labor increases by 5 units. Without plugging the new numbers for \( K \) and \( L \) into the production function, compute approximately how much the firm can now produce.

**Answer:**

   a) Maximum production is \( Q = (1000)^{1/3}(125)^{2/3} = 250 \).

   b) Now \( MP_K = \partial Q/\partial K = \frac{1}{3}K^{-2/3}L^{2/3} \) and \( MP_L = \partial Q/\partial L = \frac{2}{3}K^{1/3}L^{-1/3} \). Using \( K = 1000 \) and \( L = 125 \) yields \( MP_K = \frac{1}{12} \) and \( MP_L = \frac{4}{3} \).

   c) The change in production is \( MP_K \Delta K + MP_L \Delta L = -1/6 + 20/3 = 6.5 \). The resulting output level is 256.5.

5. Let \( f(x, y, z) = x^2 + 3xy + 4y^2 + e^z - 9 \).

   a) Find a point \((x_0, y_0, z_0)\) satisfying \( f(x_0, y_0, z_0) = 0 \).

   b) Can \( x \) be expressed as a function \( g(y, z) \) in some neighborhood of \((x_0, y_0, z_0)\)?

   c) Compute \( dg(y_0, z_0) \).
Answer:

a) The point \((x_0, y_0, z_0) = (1, 1, 0)\) satisfies the equation.

b) We compute \(\partial f/\partial x = 2x + 3y\). Plugging in \(x = 1\) and \(y = 1\), we obtain 5. The derivative is invertible (not zero). The Implicit Function Theorem then yields the desired function \(g\). In fact, it is possible to compute \(g(y, z) = [−3y + \sqrt{36 − 7y^2 − 4e^z}] / 2\).

c) The derivative is given by

\[
dg(1, 1, 0) = -\frac{1}{5} \left( \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \bigg|_{(1,1,0)}
= -\frac{1}{5} (3x + 8y, e^z) \bigg|_{(1,1,0)}
= \left( -\frac{11}{5}, -\frac{1}{5} \right).
\]