Mathematical Economics Final, December 10, 2002

1. Consider the function $f(x, y) = x^4 - 2x^2y + 2y^2 + 3$. Find and classify (maximum, minimum, saddlepoint) all critical points of f.

   Answer: The first-order conditions are

   $$0 = 4x^3 - 4xy$$
   $$0 = -2x^2 + 4y$$

   Substituting the second equation into the first shows $4x^3 - 8x^3 = 0$. This implies $x = 0$ (so $y = 0$). There is only one critical point, $(0, 0)$. The function has Hessian

   $$\begin{pmatrix} 12x^2 - 4y & -4x \\ -4x & 4 \end{pmatrix},$$

   which is positive semidefinite, but not positive definite. We can't classify $(0, 0)$ as a maximum, minimum, or saddlepoint. [However, $f = (x^2 - y)^2 + y^2 + 3$, which has a global minimum at $(0, 0)$.]

2. Consider the consumer's problem of maximizing $u(x, y)$ subject to the constraints $px + y \le I$, $x \ge 0$, and $y \ge 0$. You may presume $p > 0$, $I > 0$, $u \in C^2$, $du \gg 0$ and d^2u negative definite. Further, assume $\frac{\partial u}{\partial x}(0, y) = +\infty$ and $\frac{\partial u}{\partial y}(x, 0) = +\infty$.

   a) Find the first-order conditions for a maximum.

   Answer: Form the Lagrangian $L = u - \lambda(px + y - I) + \mu x + \nu y$. The first-order conditions are

   $$0 = \frac{\partial u}{\partial x} - \lambda p + \mu$$
   $$0 = \frac{\partial u}{\partial y} - \lambda + \nu$$

   b) Do the first-order equations completely characterize the solution? Or are additional equations required? If so, what are they?

   Answer: Additional equations are required: complementary slackness, the constraints, and non-negativity. Some simplification is possible. The first-order conditions cannot be satisfied if either $x = 0$ or $y = 0$ because the derivatives become infinite. This implies $\mu = 0$ and $\nu = 0$ by complementary slackness. Moreover, the positivity of du implies $\lambda > 0$. Complementary slackness then implies $px + y = I$. The additional equation required is $px + y = I$, which gives us three equations in three unknowns (x, y, λ).

   c) Are the second-order sufficient conditions satisfied?

   Answer: The Hessian of L is d^2u, which is negative definite, so the second-order conditions for a maximum are satisfied.

   d) Consider the system from parts (a) or (b) (as appropriate) as implicitly defining (x^*, y^*, λ^*) as functions of (p, I). Is $x^*(p, I)$ a continuously differentiable function? Why?

   Answer: The relevant system is

   $$0 = \frac{\partial u}{\partial x} - \lambda p$$
   $$0 = \frac{\partial u}{\partial y} - \lambda$$
   $$0 = px + y - I$$

If we call the right hand side $F(x, y, \lambda, p, I)$, we can apply the implicit function theorem provided $d_{(x,y,\lambda)}F$ is invertible. We have

$$d_{(x,y,\lambda)}F = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & -p \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & -1 \\ -p & -1 & 0 \end{pmatrix}.$$

The determinant is

$$2p\frac{\partial^2 u}{\partial x \partial y} - p^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = (-1, -p)d^2u\begin{pmatrix} -1 \\ -p \end{pmatrix}.$$

Since d^2u is negative definite, the determinant is negative, and so the matrix is invertible. The implicit function theorem then shows (x, y, λ) is a C^1 function of (p, I).

3. A consumer has the quasi-linear utility function $u(x, y) = x + y^2$. The consumer consumes non-negative quantities of both goods, subject to the budget constraint: $px + y \le 6$. Find (x^*, y^*) that maximizes utility subject to the above constraints. Be sure to check the constraint qualification and second-order conditions.

Answer: Note that

$$dg = \begin{pmatrix} p & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since at most two constraints can bind, constraint qualification is satisfied. The Lagrangian is $L = x + y^2 - \lambda(px + y - 6) + \mu x + \nu y$. The first-order conditions are

$$0 = 1 - \lambda p + \mu$$
$$0 = 2y - \lambda + \nu.$$

The first equation tells us that $\lambda \ge 1/p > 0$. Complementary slackness then implies $px + y = 6$. Now there are two cases to consider.

If $x = 0$, then $y = 6$. By complementary slackness, $\nu = 0$. The second first-order condition implies $\lambda = 12$. Provided $12p \ge 1$, $\mu = 12p - 1 \ge 0$, and we have a critical point $(0, 6)$.

If $y = 0$, then $x = 6/p$. By complementary slackness, $\mu = 0$. The first first-order condition implies $\lambda = 1/p$. Substituting in the second equation, we find $\nu = 1/p$. This is our second critical point, $(6/p, 0)$.

We now turn to the second-order conditions. Since there are $n = 2$ variables and $m = 2$ constraints, we have no second-order conditions available.

What we can do is compare the values of utility: $u(0, 6) = 36$ while $u(6/p, 0) = 6/p$. If $p > 1/6$, $(0, 6)$ is the maximizer while if $p < 1/6$, $(6/p, 0)$ is the maximizer. If $p = 1/6$, both are maximizers.

4. A consumer is endowed with $T > 0$ units of a resource r that may either be consumed or sold at price $w > 0$. The resource cannot be bought on the market, only sold. The consumer also consumes a consumption good c which has price $p > 0$. The budget constraint is $pc + wr \le wT$. The consumer is also subject to the constraints $c \ge 0$, $r \ge 0$, and $r \le T$ (the last constraint reflects the fact that the resource cannot be purchased). The utility function is $u(c, r) = \ln c + 2 \ln r$. Solve the consumer's problem, paying attention to constraint qualification and the second-order conditions.

Answer: The Lagrangian is $L = \ln c + 2\ln r - \lambda(pc + wr - wT) + \mu c + \nu r - \rho(r - T)$. The first-order conditions are

$$0 = \frac{1}{c} - \lambda p + \mu$$
$$0 = \frac{2}{r} - \lambda w + \nu$$

Note that neither $c = 0$ or $r = 0$ allows a solution. Thus $\mu = \nu = 0$ by complementary slackness. Moreover, $r = T$ implies $c = 0$, so it is also impossible. Thus $\rho = 0$ by complementary slackness. However, $\lambda = 1/pc > 0$, so $pc + rw = wT$ is the only binding constraint.

Constraint qualification is clearly satisfied with this one constraint. We solve the first-order equations, obtaining

$$c = \frac{2wT}{3p} \qquad \text{and} \qquad r = \frac{T}{3}.$$

Finally, we need only look at the determinant of the bordered Hessian because there is one constraint and two unknowns. The bordered Hessian is

$$H = \begin{pmatrix} 0 & p & w \\ p & -c^{-2} & 0 \\ w & 0 & -2r^{-2} \end{pmatrix}.$$

Its determinant is $2(p/r)^2 + 4(cw)^2 > 0$. Since $n = 2$, $\det H(-1)^2 > 0$, which implies we have a maximum.

5. Let A be an $n \times n$ positive definite matrix. Define $N(x) = \sqrt{x^T A x}$. Is N a norm? That is, does it obey the three conditions a norm must obey?

   Answer: It is a norm. 1) It is absolutely homogeneous of degree 1 by construction. 2) It is clearly non-negative, and if $N(x) = 0$, the quadratic form $x^T A x = 0$. Since this form is positive definite, $x = 0$. Finally, the triangle inequality must be shown.