Mathematical Economics Midterm #1, October 2, 2003

1. Let A and B be $n \times n$ matrices. Suppose that $(A-B)^2 = A^2 - 2AB + B^2$. Show that $AB = BA$.

   We compute $(A-B)^2 = (A-B)(A-B) = A(A-B) - B(A-B) = A^2 - AB - BA + B^2$. Equating to $A^2 - 2AB + B^2$, we find $A^2 - 2AB + B^2 = A^2 - AB - BA + B^2$. Canceling squared terms yields $-2AB = -AB - BA$, so $BA = AB$.

   NB. The problem does not claim $(A-B)^2 = A^2 - 2AB + B^2$ for all matrices A and B. In particular, it does not claim $(B-A)^2 = A^2 - 2BA + B^2$.

2. Let
$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

   a) Find the eigenvalues of A.
$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda - 3)(\lambda - 2).$$
   It follows that the eigenvalues are $\lambda = 2, 3$.

   b) Find an eigenvector for each eigenvalue in part (a).
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - 2I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$
   Setting $y = 1$, we obtain the eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - 3I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$
   Setting $y = 1$, we obtain the eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   c) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Write $\mathbf{x}$ as a linear combination of the eigenvectors (alternatively, find the coordinates of $\mathbf{x}$ in the basis of eigenvectors).

   We write
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\alpha - \beta \\ \alpha + \beta \end{bmatrix}.$$
   Solving for α and β we find $\alpha = -1$ and $\beta = 1$. Then $-\mathbf{v}_1 + \mathbf{v}_2$ is the required linear combination.

3. Consider the linear system
$$\begin{aligned} w + x + y + z &= 1 \\ w - x + y + z &= 1 \\ 2w - 3x + 2y + 2z &= 3. \end{aligned}$$

   a) What is the rank of the matrix of coefficients?
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & -3 & 2 & 2 \end{bmatrix}.$$

We now row-reduce

$$A \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & -5 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 2.

b) Does this system have any solutions?

The augmented matrix is

$$\hat{A} == \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & -3 & 2 & 2 & 3 \end{bmatrix}.$$

We now row-reduce

$$A \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & -5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which has rank 3. Since the rank of the augmented matrix is larger than the rank of the matrix of coefficients, the system has no solution.

c) If the system has a solution, is it unique?

The system does not have a solution.

4. Consider a modified version of the Keynesian model. We use a standard IS-curve: $sY + ar = I^o + G$, but modify the LM curve to depend on the price level P, so $M_s = M^o + mY - hr + gP$. We close the model by adding an upward sloping aggregate supply (AS) curve $Y = Y^o + bP$. You may suppose that a, b, g, h, m, s, G, I^0, M^o, M_s, and Y^o are all positive and that $M_s > M^o$.

a) This system has 3 variables: r, P, and Y. Does the system have a unique solution?

The system is

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr + gP &= M_s - M^o \\ Y - bP &= Y^o. \end{aligned}$$

The coefficient matrix is

$$A = \begin{bmatrix} s & a & 0 \\ m & -h & g \\ 1 & 0 & -b \end{bmatrix} \qquad \text{and} \qquad |A| = ag + bhs + abm > 0$$

Because the determinant is non-zero, the system has a unique solution.

b) Suppose that the IS and LM curves are as defined above. Does the IS-LM system have a unique solution for r, P, and Y?

In this case the system is

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr + gP &= M_s - M^o. \end{aligned}$$

There are fewer equations than unknowns, so it cannot have a unique solution. (Rank is 2 while there are 3 columns.)

c) Is the price level always positive? If not, what is required to make the price level positive.

We apply Cramer's rule to calculate the price level.

$$P = \frac{\begin{vmatrix} s & a & I^o + G \\ m & -h & M_s - M^o \\ -b & 1 & Y^o \end{vmatrix}}{|A|} = \frac{a(M_s - M^o - mY^o) + h(I^o + G - sY^o)}{|A|}$$

The price level is positive if $aM_s + h(I^o + G) > (am + sh)Y^o + aM^o$.

5. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

a) Is $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a linearly independent set? Justify your answer.

Let B denote the matrix of vectors. Note that $\det B = 1$. This implies that the rank is 3. Since the rank of B is equal to the number of columns, the vectors are linearly independent.

b) Does $\mathcal{B}$ span $\mathbb{R}^3$? Justify your answer.

Since the rank of B is 3, which is the number of rows, the vectors span $\mathbb{R}^3$.

c) Is $\mathcal{B}$ a basis for $\mathbb{R}^3$? If so, express $\mathbf{y} = (5, 3, 1)'$ in terms of the vectors in $\mathcal{B}$.

The set $\mathcal{B}$ is a basis because it is a linearly independent set that spans $\mathbb{R}^3$. Now

$$B^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The coordinates of $\mathbf{y}$ are:

$$B^{-1}\mathbf{y} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -4 \end{bmatrix}.$$

Thus $\mathbf{y} = 11\mathbf{x}_1 - 2\mathbf{x}_2 - 4\mathbf{x}_3$.