Mathematical Economics Midterm #2, November 6, 2003

1. Are the following sets open in $\mathbb{R}^2$? Closed? Neither? Explain.

   a) $A = \{(x,y) : x^2 + y^2 = 1\}$.

   Answer: Let $f(x,y) = x^2 + y^2$. Because f is a quadratic polynomial, it is continuous. Now $A = f^{-1}(\{1\})$ and so A is closed because it is the inverse image of a closed set. However, A is not open. To see this consider $B_\varepsilon(1,0)$. This contains points such as $(1, \varepsilon/2)$ that are not in the set A. Since no ball about $(1,0) \in A$ is contained in A, A is not open.

   b) $B = \{(x,y) : x^2 + 2xy + 12y^2 \leq 36\}$.

   Answer: Let $f(x,y) = x^2 + 2xy + 12y^2$. Because f is a quadratic polynomial, it is continuous. Now $B = f^{-1}((-\infty, 36])$ and so B is closed because it is the inverse image of a closed set. However, B is not open. To see this consider $B_\varepsilon(6,0)$. This contains points such as $(6, \varepsilon/2)$ that are not in the set B. Since no ball about $(6,0) \in B$ is contained in B, B is not open.

2. Consider the sequence $x_n = (-1)^n + n^2/(n^3 + 2n + 1)$.

   a) Does it converge? If so, what is its limit?

   Answer: It does not converge. Note that $|x_n - x_{n+1}| \to 2$.

   b) If it doesn't converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   Answer: Yes, it has convergent subsequences. Two obvious ones are the subsequence of even terms (with limit 1) and the subsequence of odd terms (with limit -1).

3. Consider the differential system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x + 3y \end{pmatrix}$$

   with initial conditions $x(0) = 2$ and $y(0) = 3$.

   a) Write the system in matrix form.

   Answer:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

b) Find the eigenvalues of this system.

Answer: The eigenvalue equation is $0 = (3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8$. This has solutions $\lambda = 2$ and $\lambda = 4$.

c) Is this system stable?

Answer: No, both eigenvalues are positive so it is unstable.

d) Find the solution to this system.

Answer: Since the original matrix is symmetric, we can find orthonormal eigenvalues. One eigenvector corresponding to $\lambda = 4$ is $(1,1)'$ and an eigenvector corresponding to $\lambda = 2$ is $(1,-1)'$. In both cases we can normalize by dividing by $\sqrt{2}$ to obtain orthonormal eigenvectors.

This yields basis matrix:

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{with} \qquad B^{-1} = B' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

and then

$$B^{-1}AB = \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

The solution is

$$B^{-1}e^{\Lambda t}B\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5e^{4t} - e^{2t} \\ 5e^{4t} + e^{2t} \end{pmatrix}.$$

4. Robinson Crusoe has utility function $u(x,y) = \ln x + xy + y^2$. Crusoe has a production possibilities set given by $\{(x,y) : x \geq 1, y \geq 0, x + 2y \leq 4, 3x + y \leq 9\}$.

a) Is the production possibility set a closed set? Is it a bounded set? Explain.

Answer: Yes, the production possibility set is closed. It can be written as $\{(x,y) : x \geq 1\} \cap \{(x,y) : y \geq 0\} \cap \{(x,y) : x + 2y \leq 4\} \cap \{(x,y) : 3x + y \leq 9\}$. Each of these 4 sets is the inverse image of a closed set under a linear function and is closed. Thus the intersection is also closed.

The production possibilities set is also bounded. To see this, note that $1 \leq x \leq 3$ and $0 \leq y \leq 2$. It follows that $\|(x,y)\|^2 \leq 3^2 + 2^2 = 13$, so $\|(x,y)\| \leq \sqrt{13}$.

b) Show that Crusoe's problem of maximizing utility over his production set has a solution.

Answer: The production set is compact because it is closed and bounded. Because $x \geq 1$, the function $\ln x$ is continuous on the production set. The polynomial terms are also continuous, so the sum f is continuous. By the Weierstrass Theorem, this continuous function must have a maximum on the compact production possibilities set.

5. Let $f(x, y) = x^2/(x^2 + y^2)$ so that $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$.

   a) What is the range of f?

   Answer: Clearly $0 \leq f(x, y) \leq 1$. Both endpoints can occur: $f(0,1) = 0$ and $f(1,0) = 1$. The range is $[0,1]$.

   b) Is f onto?

   Answer: No, $\operatorname{ran} f \neq \mathbb{R}^2$.

   c) Is f one-to-one?

   Answer: No, $f(0,1) = f(0,1/2) = 0$.

   d) Is f continuous?

   Answer: Yes. It is the quotient of continuous functions and the denominator is non-zero.

   e) Compute df

   Answer:

$$df = \left(\frac{2xy^2}{(x^2+y^2)^2}, \frac{-2yx^2}{(x^2+y^2)^2} \right).$$