Homework Assignment #4

13.12 Write the following quadratic forms in matrix form:

a) $x_1^2 - 2x_1x_2 + x_2^2$.

b) $5x_1^2 - 10x_1x_2 - x_2^2$.

c) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$.

Answer: If we require the matrices to be symmetric, the solutions are:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 5 & -5 \\ -5 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{bmatrix}.$$

13.21 Let $f\colon \mathbb{R}^k \to \mathbb{R}^1$ be continuous at the point $\mathbf{a} = (a_1, \ldots, a_k)$. Consider the function $g\colon \mathbb{R}^1 \to \mathbb{R}^1$ defined by $g(t) = f(t, a_2, \ldots, a_k)$. Show that g is continuous at a_1. This result implies that if f is continuous, its restriction to any line parallel to a coordinate axis is also continuous. However, the converse is not true. Consider the function $f(x, y) = xy^2/(x^2 + y^4)$. Show that $f_1(t) = f(t, a)$ and $f_2(t) = f(a, t)$ are continuous functions of t for each fixed a. Show that f itself is not continuous at $(0, 0)$. [Hint: Take a sequence on the diagonal.]

Answer: Let $t_n \to t$. Then $(t_n, a_2, \ldots, a_k) \to (t, a_2, \ldots, a_k)$. Since f is continuous, $g(t_n) = f(t_n, a_2, \ldots, a_k) \to f(t, a_2, \ldots, a_k) = g(t)$, showing continuity of g.

For the function $f(x, y) = xy^2/(x^2 + y^4)$ we define $f(0, 0) = 0$. Then $f_1(t) = ta^2/(t^2 + a^4)$. Clearly $f_1(t_n) \to f_1(t)$ whenever $t_n \to t$, even if $t = 0$. When $a = 0$, it reduces to 0 for $t \neq 0$. Since we have set $f(0, 0) = 0$, this is also continuous. The case of $f_2(t) = at^2/(a^2 + t^4)$ is similar.

The hint is not quite correct as one should use a parabola, not a line. Consider $f_3(t) = f(t^2, t) = 1/2$. This does not converge to 0 as $t \to 0$, so f is not continuous.

14.7 A firm has the Cobb-Douglas production function $y = 10x_1^{1/3}x_2^{1/2}x_3^{1/6}$. Currently, it is using the input bundle $(27, 16, 64)$.

a) How much is it producing?

b) Use differentials to approximate its new output when x_1 increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same.

c) Use a calculator to compare your answer in part *b* with the actual output.

d) Do *b* and *c* for $\Delta x_1 = \Delta x_2 = 0.2$ and $\Delta x_3 = -0.4$.

Answer:

a) The firm is currently producing $10 \times 27^{1/3}16^{1/2}64^{1/6} = 10(3)(4)(2) = 240$.

b) Here $df(\mathbf{x}) = f(\mathbf{x})(1/3x_1, 1/2x_2, 1/6x_3)$, so $df(27, 16, 64) = 240(1/81, 1/32, 1/384) = (2.963, 7.5, 0.625)$. The new output is approximately $f(27, 16, 64) + (0.1, -0.3, 0)df = 240 + 0.2963 - 2.25 = 238.0463$.

c) The calculator gives 238.0325, which is slightly lower, an error of about 0.006%.

d) Here the new value is estimated at $240 + (2.963, 7.5, 0.625) \cdot (0.2, 0.2, -0.4) = 240 + 0.5926 + 1.5 - 0.25 = 241.8426$. The actual value $f(27.2, 16.2, 63.6)$ is about 241.837, a difference of about 0.002%.

14.12 At a given moment in time, the marginal product of labor is 2.5 and the marginal product of capital is 3, the amount of capital is increasing by 2 each unit of time and the rate of change of labor is +0.5. What is the rate of change of output?

Answer: The rate of change of output is $2.5 \times 0.5 + 3 \times 2 = 1.25 + 6 = 7.25$

14.26 For what values of their parameters do Cobb-Douglas and CES productions functions obey the law of diminishing marginal returns?

Answer: We consider these functions for $\mathbb{R}^2_+$, where the Cobb-Douglas production function is $f(x_1, x_2) = kx_1^{b_1}x_2^{b_2}$ and the CES production function is $g(x_1, x_2) = k(c_1x_1^{-a} + c_2x_2^{-a})^{-b/a}$.

We first require that production be positive ($f > 0$ and $g > 0$) and that marginal products be positive ($\partial f/\partial x_i > 0$ and $\partial g/\partial x_i > 0$).

This implies $k > 0$, $b_1 > 0$ and $b_2 > 0$ for the Cobb-Douglas function and that $k > 0$, $c_1 > 0$, $c_2 > 0$ and $b > 0$ for the CES function.

For Cobb-Douglas, diminishing marginal returns requires that $\partial^2 f/\partial x_i^2 < 0$ for $i = 1, 2$. Now $\partial^2 f/\partial x_1^2 = kb_1(b_1 - 1)x_1^{b_1-2}x_2^{b_2}$ and $\partial^2 f/\partial x_2^2 = kb_2(b_2 - 1)x_1^{b_1}x_2^{b_2-2}$. The required conditions are $b_1 < 1$ and $b_2 < 1$.

For CES, diminishing marginal returns requires that $\partial^2 g/\partial x_i^2 < 0$ for $i = 1, 2$. Now $\partial g/\partial x_i = bc_ikx_i^{-a-1}(c_1x_1^{-a} + c_2x_2^{-a})^{-(a+b)/a}$. The second derivatives are

$$\frac{\partial^2 g}{\partial x_i^2} = bkc_ix_i^{-2a-2}(c_1x_1^{-a} + c_2x_2^{-a})^{-(2a+b)/a}\left[c_i(b-1) - (a+1)c_j\left(\frac{x_i}{x_j}\right)^a\right].$$

Thus $b < 1$ and $a > -1$ are the required conditions for the CES function.