Mathematical Economics Exam #1, September 26, 2011

1. In $\mathbb{R}^3$,

   a) Find all vectors that are perpendicular to

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

   Answer: These vectors $\mathbf{z}$ obey $\mathbf{x}_1 \cdot \mathbf{z} = 0$ and $\mathbf{x}_2 \cdot \mathbf{z} = 0$. In matrix form, this becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}.$$

   Then z_3 is the only free variable and the perpendicular vectors are all vectors $\mathbf{z}$ obeying $z_1 = z_2 = -z_3/2$

   b) Do the vectors described in part (a) form a vector subspace of $\mathbb{R}^3$? Justify your answer.

   Answer: Yes, they form a vector space. Let $V = \{\mathbf{z} : \mathbf{z} \cdot \mathbf{x}_i = 0 \text{ for } i = 1, 2\}$. Suppose $\mathbf{y}, \mathbf{z} \in V$. Then $\mathbf{x}_i \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x}_i \cdot \mathbf{y} + \mathbf{x}_i \cdot \mathbf{z} = 0$ for $i = 1, 2$, so $(\mathbf{y} + \mathbf{z}) \in V$. Also, if α is a real number $\mathbf{x}_i \cdot (\alpha \mathbf{z}) = \alpha \mathbf{x}_i \cdot \mathbf{z} = 0$. This shows that both scalar multiples and vector sumes of vectors in V are in V, establishing that it is a vector subspace of $\mathbb{R}^3$.

2. Consider the linear system

$$\begin{aligned} w + \ x + \ y + \ z &= 1 \\ w - \ x + \ y + \ z &= 1 \\ 2w - 3x + 2y + 2z &= 2. \end{aligned}$$

   a) What is the rank of the matrix of coefficients?

   Answer: The rank is 2. This is clear once we row-reduce the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & -3 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{(2)-(1),(3)-2(1)} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{(3)-5/2(2)} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   b) Does this system have any solutions?

   Answer: Yes.

   c) If the system has one or more solutions, describe them. Is the solution unique?

   Answer: Any vector of the form $(1 - y - z, 0, y, z)^T$ will satifisy the system. The solution is not unique.

3. Consider the set $T = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\} \subset \mathbb{R}^3$.

   a) Is the set T open, closed, or neither when considered as a subset of $\mathbb{R}^3$?

   Answer: The set is closed.

   b) Prove your answer in part (a).

   Answer: One way to prove this is to note that $f(\mathbf{x}) = \|\mathbf{x}\|$ is continuous, so if $\mathbf{x}_n \to \mathbf{x}$, then $\|\mathbf{x}_n\| \to \|vx\|$. Since $\|\mathbf{x}_n\| = 1$, $\|\mathbf{x}\| = 1$ and $\mathbf{x} \in T$.

   If you don't know that the norm is continuous, use the triangle inequality to show $\big|\, \|\mathbf{x}_n\| - \|\mathbf{x}\| \,\big| \le \|\mathbf{x}_n - \mathbf{x}\|$, which can be used to show that the norm is continuous.

   Another method is to consider $T^c = B_1(\mathbf{0}) \cup \{\mathbf{x} : \|\mathbf{x}\| > 1\}$ and show it is open. Since the open ball of radius 1 is open and the union of open sets is open, this reduces to showing $\{\mathbf{x} : \|\mathbf{x}\| > 1\}$ is open.

4. Consider the following set of vectors in $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

   a) Is $\mathcal{B}$ a linearly independent set? Justify your answer.

   Answer: No. No more than three vectors can be linearly independent in $\mathbb{R}^3$.

   b) Does $\mathcal{B}$ span $\mathbb{R}^3$? Justify your answer.

   Answer: Yes, it spans $\mathbb{R}^3$. In fact, the first three elements of $\mathcal{B}$ span $\mathbb{R}^3$. This can be seen by calculating

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1.$$

   Since the determinant is non-zero, the three vectors span $\mathbb{R}^3$.

   c) Is $\mathcal{B}$ a basis for $\mathbb{R}^3$?

   Answer: No. It is not a linearly independent set.