Mathematical Economics Exam #2, November 9, 2011

1. Let $f(x,y,z) = (x+y)z+y$. The point $(1,1,0)$ satisfies the equation $f(x,y,z) = 1$. Can x be written as a C^1 function of (y,z) near $(1,1,0)$? If so, let g be the function with $f(g(y,z),y,z) = 1$ and find $dg(1,0)$.

   Answer: We attempt to apply the Implicit Function Theorem, but calculating $\partial f/\partial x = z$, we find $\partial f/\partial x(1,1,0) = 0$. In fact, if we attempt to solve $f(x,y,z) = 1$ for x, we obtain $x = (1-y-yz)/z$, which is not defined at $z = 0$. So x cannot be written as a C^1 function of (y,z) near $(1,1,0)$.

2. Suppose $f(x_1,x_2,x_3) = \arcsin(x_1x_2x_3^{-2})$.

   a) Is f a homogeneous function on $\mathbb{R}^3_{++}$? If so, what is the degree of homogeneity?

   b) Is it a homothetic function?

   c) Suppose you maximize f over the budget set determined by $\mathbf{p} = (1,1,1)$ and income I. Describe how the maximum point $\mathbf{x}(I)$ varies as I varies.

   Answer: My apologies for the error in this question. The function arcsin is only defined for numbers in $[-1,1]$. It may not be defined over the entire budget set, and that is not the problem I intended. I should have used arctan instead, which is defined for all real numbers. As to why I would want to use such a weird function, it's to try to discourage solution via computation.

   a) For $\lambda > 0$, $f(\lambda x_1, \lambda x_2, \lambda x_3) = \arcsin(\lambda^2 x_1 x_2 \lambda^{-2} x_3^{-2}) = \arcsin(x_1x_2x_3^{-2})$, showing that f is homogeneous of degree 0.

   b) Any homogeneous function is homothetic, so f is homothetic.

   c) We modify the question so it has a solution by adding the additional constraint that $x_1x_2 \le x_2^2$. This restricts our attention to the portion of the budget set where $f(\mathbf{x})$ is defined.

   The homogeneity of degree zero implies that $\mathbf{x}$ and $\lambda\mathbf{x}$ for $\lambda > 0$ yield the same utility. This implies that if $\mathbf{x}$ is optimal, so $\lambda\mathbf{x}$ for $0 < \lambda < 1$. It follows that there are many optima. Let $X(\mathbf{p}, I)$ be the set of optima. If income changes from I to I', the set $(I'/I)X(\mathbf{p}, I)$ is now feasible and optimal. Thus $(I'/I)X(\mathbf{p}, I) \subset X(\mathbf{p}, I')$. Running the argument the other way, we find $(I'/I)X(\mathbf{p}, I) = X(\mathbf{p}, I')$. Setting $I' = 1$, we find $X(\mathbf{p}, I) = IX(\mathbf{p}, 1)$. I.e., the set of solutions is linear in I. Note that this is basically the same argument given in class, and you may reference that to solve this problem.

3. Consider the quadratic form $Q(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_1x_3 - 2x_2^2 + 6x_2x_3 + 3x_3^2$. Does Q have a maximum, minimum, or saddlepoint at $(0,0,0)$ subject to the constraint $x_2 - x_3 = 0$?

   Answer: The quadratic form can be generated by the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 3/2 \\ 1 & -2 & 3 \\ 3/2 & 3 & 3 \end{bmatrix}.$$

   We must examine the bordered Hessian

$$H = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 3/2 \\ 1 & 1 & -2 & 3 \\ -1 & 3/2 & 3 & 3 \end{bmatrix}.$$

   With $n = 3$ variables and $k = 1$ constraint, we must check the last $n - k = 2$ leading principal minors of H. They are $H_3 = -1$ and $H_4 = -3/4$. Since the minors are the same sign and $(-1)^k H_4 > 0$, we have a constrained minimum.

   An alternate (and quicker!) solution is to subsitute the constaint $x_3 = x_2$ into the form, yielding $R(x_1, x_2) = x_1^2 + 5x_1x_2 + 7x_2^2$. This form has Hessian $M = \begin{bmatrix} 1 & 5/2 \\ 5/2 & 7 \end{bmatrix}$. Because $M_1 = 1$ and $M_2 = 3/4$, it is positive definite and $\mathbf{0}$ is a minimum.

4. Utility on $\mathbb{R}_+^2$ is given by $u(x, y) = x + \sqrt{y}$. Find all the points that maximize utility over the set $B = \{(x, y) : 2x + y \leq 1,\ x + 2y \leq 1, \text{ and } x, y \geq 0\}$. Don't forget to check the constraint qualification conditions.

   Answer: The derivative of the vector of constraint functions is:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Each row is non-zero, so if only one constraint binds, the NDCQ condition holds. Each of the 6 second-order submatrices has non-zero determinant, so if exactly two constraints bind, the NDCQ condition holds. Since it is not possible for three or more constraints to simultaneously bind, NDCQ is satisfied in all cases.

   The Lagrangian is $x + \sqrt{y} - \lambda_1(2x + y - 1) - \lambda_2(x + 2y - 1) + \mu_1 x + \mu_2 y$. The resulting first-order conditions are

$$1 + \mu_1 = 2\lambda_1 + \lambda_2$$

and

$$\frac{1}{2}y^{-1/2} + \mu_2 = \lambda_1 + 2\lambda_2.$$

Note that y cannot be zero, so $\mu_2 = 0$ by complementary slackness. Since $1 \le 2\lambda_1 + \lambda_2$, at least one of the λ's is positive. The corresponding constraint binds by complementary slackness.

There are three cases:

i) If both λ constraints bind, $(x, y) = (1/3, 1/3)$. In that case the μ_i's are both zero, so $1 = 2\lambda_1 + \lambda_2$ and $\sqrt{3}/2 = \lambda_1 + 2\lambda_2$. This has solution $\lambda_1 = (4 - \sqrt{3})/6 > 0$ and $\lambda_2 = (\sqrt{3} - 1)/3 > 0$. It is a potential maximum

ii) If $2x + y < 1$, $\lambda_1 = 0$ by complementary slackness, so we have $1 + \mu_1 = \lambda_2$ and $\frac{1}{2}y^{-1/2} = 2\lambda_2$. If $x > 0$, $\mu_1 = 0$ by complementary slackness, and $y = 1/16$. Since $x + 2y = 1$, $x = 7/8$. But then $2(7/8) + 1/16 > 1$, contradicting the assumption that $2x + y < 1$. If $x = 0$, the λ_2 constraint yields $y = 1/2$. Then $\lambda_2 = 1/2\sqrt{2}$. But then $\lambda_2 < 1 + \mu_1$, so it cannot be optimal. Either way, there is no solution in this case.

iii) , If $x + 2y < 1$, $\lambda_2 = 0$ by complementary slackness, so we have $1 + \mu_1 = 2\lambda_1$ and $\frac{1}{2}y^{-1/2} = \lambda_1$. If $x = 0$, the constraint $2x + y = 1$ yields $y = 1$. But this violates the constraint that $x + 2y < 1$. What if $x > 0$? Then $\mu_1 = 0$ by complementary slackness. In that case $\lambda_1 = 1/2$ and $y = 1$ and so $x = 0$, contradicting $x > 0$ (and also violating $x + 2y < 1$ again).

Thus $(1/3, 1/3)$ is the only possible maximum. Since there is a maximum, this must be it (we don't have to check second-order conditions).