# Mathematical Economics Final, December 7, 2011

1. Let $\mathbf{p} \gg \mathbf{0}$ and $m > 0$. Let $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{p}\cdot\mathbf{x} \le m\}$.

   *a*) Is $B(\mathbf{p}, m)$ a closed set? Justify your answer.

   *b*) Is $B(\mathbf{p}, m)$ a bounded set? Justify your answer.

   *c*) If $u$ is a continuous function from $\mathbb{R}^2_+$ into $\mathbb{R}$, does it have a maximum over $B(\mathbf{p}, m)$? Why?

   **Answer:** This question has been corrected so that $B(\mathbf{p}, m) \subset \mathbb{R}^2_+$ instead of leaving it unbounded below. Many of you answered the intended version.

   *a*) Yes. The set $B(\mathbf{p}, m)$ is the intersection of $B_1 = \{\mathbf{x} : x_1 \ge 0\}$, $B_2 = \{\mathbf{x} : x_2 \ge 0\}$, and $B_3 = \{\mathbf{x} : \mathbf{p}\cdot\mathbf{x} \le m\}$. Let $f_1(\mathbf{x}) = x_1$, $f_2(\mathbf{x}) = x_2$, and $f_3(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x}$. Each of these functions is linear, and so continuous. Now $B_1 = f_1^{-1}[0, +\infty)$, $B_2 = f_2^{-1}[0, +\infty)$, and $B_2 = f_3^{-2}(-\infty, m]$. Since each $B_i$ is the inverse image of a closed set, each $B_i$ is closed. As the intersection of three closed sets, $B(\mathbf{p}, m)$ is also closed.

   *b*) Yes. If $\mathbf{x} \in B(\mathbf{p}, m)$, $x_i \ge 0$ and $p_1x_1 + p_2x_2 \le m$. It follows that $p_1x_1 \le m$ and $p_2x_2 \le m$. Let $p = \min(p_1, p_2) > 0$. Then $0 \le x_i \le m/p$, implying that $B(\mathbf{p}, m)$ is bounded.

   *c*) Yes. Since $B(\mathbf{p}, m)$ is both closed and bounded, it is compact. The Weierstrass Theorem then tells us that any continuous function has a maximum over the compact set $B(\mathbf{p}, m)$.

2. On $\mathbb{R}^2_+$, let $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$. Suppose $\mathbf{p} \gg \mathbf{0}$ and $m > 0$. Find the Marshallian demand function $\mathbf{x}(\mathbf{p}, m)$ and the indirect utililty function $V(\mathbf{p}, m)$. Don't forget to check constraint qualification and the second-order conditions.

   **Answer:** We start with the second order conditions. The Hessian of $u$ is

$$H = \begin{pmatrix} -\frac{1}{4}x_1^{-3/2} & 0 \\ 0 & -\frac{1}{2}x_2^{-3/2} \end{pmatrix}.$$

   It is clear that $H_1 < 0$ and $H_2 > 0$, implying $H$ is negative definite. The second-order conditions are satisfied.

   The constraints are $g_0(\mathbf{x}) = p_1x_1 + p_2x_2 \le m$, $g_1(\mathbf{x}) = -x_1 \le 0$, and $g_2(\mathbf{x}) = -x_2 \le 0$.

The derivative of the constraint matrix is

$$D\mathbf{g} = \begin{pmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since each row is non-zero, NDCQ holds if only one constraint binds. Since any pair of the rows is linearly independent, NDCQ holds if exactly two constraints bind. Finally, it is impossible for all three constraints to simultaneouly bind since $\mathbf{p} \gg \mathbf{0}$ and $m > 0$. The last is a good thing, as $D\mathbf{g}$ cannot have rank 3! NDCQ holds in all cases.

We turn to the first-order conditions. The Lagrangian is $\mathcal{L} = \sqrt{x_1} + \sqrt[3]{x}_2 - \lambda(p_1 x_1 + p_2 x_2 - m) + \mu_1 x_1 + \mu_2 x_2$. The resulting first-order conditions are

$$\frac{1}{2} x_1^{-1/2} = \lambda p_1 - \mu_1, \text{ and}$$
$$\frac{1}{2} x_2^{-1/2} = \lambda p_2 - \mu_2.$$

If either $x_1 = 0$ or $x_2 = 0$, we have no solution, so $x_1 > 0$ and $x_2 > 0$. By complementary slackness, $\mu_1 = 0$ and $\mu_2 = 0$.

The first-order conditions may now be rewritten as $\frac{1}{2}x_1^{-1/2} = \lambda p_1$ and $\frac{1}{2}x_2^{-1/2} = \lambda p_2$. Eliminating the $\lambda$ and rearranging, we obtain $p_2 x_2^{1/2} = p_1 x_1^{1/2}$, or $x_2 = (p_1/p_2)^2 x_1$. Then we use the budget constraint (which binds by complementary slackness), to find

$$\mathbf{x}(\mathbf{p}, m) = \left( \frac{m p_2}{p_1(p_1 + p_2)}, \frac{m p_1}{p_2(p_1 + p_2)} \right).$$

Finally, we substitute back in the utility function to obtain indirect utility,

$$V(\mathbf{p}, m) = \sqrt{\frac{m(p_1 + p_2)}{p_1 p_2}}.$$

3. Suppose the expenditure function is $e(\mathbf{p}, \bar{u}) = 2\bar{u}\sqrt{p_1 p_2}$.
   *a*) Find the Hicksian (compensated) demand function $\mathbf{h}(\mathbf{p}, \bar{u})$.
   *b*) Write $p_1/p_2$ as a function of $h_1$, and separately as a function of $h_2$.
   *c*) Use the equations derived in (b) to find a relationship between $h_1$, $h_2$, and $\bar{u}$. Infer the utility function from this relationship.

**Answer:**

*a*) We use the Shephard-McKenzie Lemma to find

$$\mathbf{h} = D_{\mathbf{p}}e = \bar{u}\left(\sqrt{\frac{p_2}{p_1}}, \sqrt{\frac{p_1}{p_2}}\right).$$

*b*) Here $p_1/p_2 = \bar{u}^2/h_1^2$ and $p_1/p_2 = h_2^2/\bar{u}^2$.

*c*) Equating the two expressions for $p_1/p_2$ yields $\bar{u}^4 = h_1^2 h_2^2$ or $\bar{u} = \sqrt{h_1 h_2}$. Since $\mathbf{h}(\mathbf{p}, \bar{u})$ always gives utility $\bar{u}$, we infer $u(\mathbf{x}) = \sqrt{x_1 x_2}$.

4. Let $A = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$. Find $\sqrt{A}$.

**Answer:** The characteristic equation is $\lambda^2 - 5\lambda + 4 = 0$, yielding eigenvalues $\sigma(A) = \{4, 1\}$. The corresponding eigenvalues are $\mathbf{v}_4 = (1, 2)^T$ and $\mathbf{v}_1 = (1, 1)$. Let $P = [\mathbf{v}_4, \mathbf{v}_1]$. We have

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

This implies $D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = P^{-1}AP$. Then $\sqrt{A} = P\sqrt{D}P^{-1}$. Since $\sqrt{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\sqrt{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

5. Consider the second-order differential equation $2\ddot{x} + 9\dot{x} - 5x = 0$.

*a*) Rewrite the differential equation as a first-order differential system.

*b*) Find the eigenvalues of the first-order system in (a).

*c*) Find the corresponding eigenvectors, then sketch the behavior of the system near the steady state ($\mathbf{0}$).

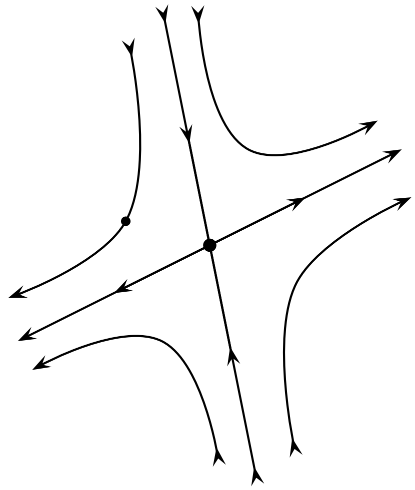
**Answer:**

*a*) Set $y = \dot{x}$. Then we can write the system as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{5}{2} & -\frac{9}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

*b*) The eigenvalues must obey $\lambda^2 + \frac{9}{2}\lambda - \frac{5}{2} = 0$. This is easily solved, yielding spectrum $\sigma = \{\frac{1}{2}, -5\}$.

*c*) The corresponding eigenvectors are scalar multiples of $\mathbf{v}_{1/2} = (2, 1)^T$ and $\mathbf{v}_{-5} = (1, -5)^T$. This is a saddlepoint with the stable axis through $\mathbf{v}_{-5}$ and the unstable axis through $\mathbf{v}_{1/2}$. See the diagram.



The stable eigenspace runs NW-SE. The lines illustrate the flow about the steady state