Mathematical Economics Exam #1, September 26, 2012

1. Let $\mathcal{Z} = \{\mathbf{z}^1, \ldots, \mathbf{z}^m\}$ be a collection of vectors in $\mathbb{R}^n$. Define $W = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\cdot\mathbf{z}^i = 0,$ for all $\mathbf{z}^i \in \mathcal{Z}\}$.

   a) Show that W is not empty.

   Answer: The vector $\mathbf{0}$ is in W because $\mathbf{0}\cdot\mathbf{z} = 0$ for every vector $\mathbf{z}$, including all those in $\mathcal{Z}$.

   b) Is W a vector subspace of $\mathbb{R}^n$? Does your answer depend on whether $m \geq n$? Justify your answers either by proof or counter-example.

   Answer: Yes, it is a vector subspace. Suppose $\mathbf{x}, \mathbf{y} \in W$. Then $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}^i = \mathbf{x}\cdot\mathbf{z}^i + \mathbf{y}\cdot\mathbf{z}^i = 0$ for every $\mathbf{z}^i \in \mathcal{Z}$ because $\mathbf{x}, \mathbf{y} \in W$. This shows $(\mathbf{x}+\mathbf{y}) \in W$. Further, for $\alpha \in \mathbb{R}$, $(\alpha\mathbf{x})\cdot\mathbf{z}^i = \alpha(\mathbf{x}\cdot\mathbf{z}^i) = 0$ for every $\mathbf{z}^i \in \mathcal{Z}$ because $\mathbf{x} \in W$. This shows $(\alpha\mathbf{x}) \in W$. Since the sum of any two vectors in W is in W and any scalar multiple of a vector is W is in $W \subset \mathbb{R}^n$, W is a vector subspace of $\mathbb{R}^n$. Note that the answer does not depend on whether $m \geq n$.

2. Consider the set $S = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| < 1, \mathbf{x} \neq \mathbf{0}\} \subset \mathbb{R}^3$.

   a) Is the set S open, closed, or neither when considered as a subset of $\mathbb{R}^3$?

   Answer: The set is open.

   b) Prove your answer in part (a).

   Answer: One way to prove this is to note that S is the intersection of two open sets, the open ball $B_1(\mathbf{0})$ and $T = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \neq \mathbf{0}\}$. We saw that $B_1(\mathbf{0})$ was open in class. The set T is open as the complement of the closed set $\{\mathbf{0}\}$ (we established in class that all singletons are closed). As the intersetion of two open sets, S is open.

   Another method is to consider any $\mathbf{y} \in S$ and show that $B_r(\mathbf{y}) \subset S$ whenever $r < \min\{1 - \|\mathbf{y}\|, \|\mathbf{y}\|\}$. Note that $\mathbf{0} \notin B_r(\mathbf{y})$ and, as in class, $B_r(\mathbf{y}) \subset B_1(\mathbf{0})$. This implies $B_r(\mathbf{y}) \subset S$, so S is open.

3. Suppose an $m \times n$ matrix A has a left inverse A^L. I.e., $A^L \times A$ is an identity matrix.

   a) How many rows and columns does A^L have? What about $A^L \times A$?

   Answer: In order to carry out the multiplication, A^L must have m columns. Now suppose A^L is $k \times m$. Then $A^L \times A$ is $k \times n$. As this is an identity matrix, it must be square, so $k = n$. It follows that A^L has n rows and m columns.

b) Does the system $A\mathbf{x} = \mathbf{b}$ always have a solution? Explain.

Answer: No. If $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, it has left inverse $A^L = (1\ 0)$. The equation $Ax = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has no solutions.

c) How are m and n related? Or can they be completely arbitrary.

Answer: If the equation $A\mathbf{x} = \mathbf{b}$ has a solution, we can premultiply by A^L to find $(A^L \times A)\mathbf{x} = A^L\mathbf{b}$. As $A^L \times A$ is the identity, this means $\mathbf{x} = A^L\mathbf{b}$. The solution is unique.

The fact that a solution is unique if it exists means that the rank of A must equal the number of columns (n). Since the rank is never more than the number of rows (m), we must have $m \geq n$.

d) How many solutions does the system $A\mathbf{x} = \mathbf{0}$ have?

Answer: This always has at least one solution, $\mathbf{x} = \mathbf{0}$. Since A^L exists, we can premultiply by A^L to find $\mathbf{x} = A^L\mathbf{0} = \mathbf{0}$. In other words, the solution is unique.

4. Let $A = \begin{bmatrix} 4 & 3 \\ 4 & 0 \end{bmatrix}$.

a) Find all numbers λ so that $A - \lambda I$ is not invertible.

Answer: $A - \lambda I$ is not invertible if and only if its determinant is zero. Now $0 = \det(A - \lambda I) = (4 - \lambda)(-\lambda) - 12 = \lambda^2 - 4\lambda - 12$ Either $\lambda = -2$ or $\lambda = 6$

b) For each λ you found in (a), find a non-zero vector $\mathbf{x}_\lambda$ obeying $A\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$.

Answer: First we set $\lambda = -2$. Then $A - (-2I) = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$. This has rank 1 and any non-zero multiple of $(1, -2)$ can be used as $\mathbf{x}_{-2}$.

The second case is $\lambda = 6$ when $A - 6I = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$. Any non-zero multiple of $(3, 2)$ can be used as $\mathbf{x}_6$.