Mathematical Economics Exam #2, November 9, 2012

1. [Corrected] Let $f(x,y,z) = x^2z^2 + 2xy + 3x - 5$. The point $(1,1,0)$ satisfies the equation $f(x,y,z) = 0$. Can x be written as a C^1 function of (y,z) near $(1,1,0)$? If so, let g be the function with $f(g(y,z),y,z) = 0$ and find $dg(1,0)$. If not, describe the cause of the problem (that a particular theorem doesn't apply here is not enough).

   Answer: Here $\partial f/\partial x = 2xz^2 + 2y + 3$ which is 5 at $(1,1,0)$. Since the derivative is non-zero, we may apply the Implicit Function Theorem to obtain g in a (y,z)-neighborhood of $(1,0)$. The Implicit Function Theorem also states that

$$dg(1,0) = \frac{-1}{\partial f/\partial x(1,1,0)} df_{(y,z)}(1,1,0).$$

   Now $df_(y,z)(1,1,0) = (2,0)$ and $\partial f/\partial x(1,1,0) = 5$, so $dg(1,0) = (-2/5, 0)$.

   Alternatively, one could solve the quadratic for x, in which case $g(y,z) = \big(-(2y+3) \pm \sqrt{(2y+3)^2 + 20z^2}\big)/2z^2$. In order to have $g(1,0) = 1$, we must take the positive sign, so

$$g(y,z) = \begin{cases} \big(-(2y+3) \pm \sqrt{(2y+3)^2 + 20z^2}\big)/2z^2 & \text{for } z \neq 0 \\ 5/(2y+3) & \text{for } z = 0. \end{cases}$$

2. Utility on $\mathbb{R}^2_+$ is given by $u(x,y) = x^\alpha y^{1-\alpha}$ where $0 < \alpha < 1$. Find all the points that maximize utility over the set $B = \{(x,y) : x^2 + y^2 \leq 1, \text{ and } x, y \geq 0\}$. Don't forget to check the both the constraint qualification and second order conditions.

   Answer: We can apply any of several constraint qualification tests. Slater's condition applies. The NDCQ condition also holds as the contraint functions have derivative

$$dg = \begin{bmatrix} 2x & 2y \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   If any one constraint binds, its row is non-zero. If any two constraints bind, the matrix of those two constaints has rank 2. All three constraints cannont simultaneously bind.

   We now form the Lagrangian $\mathcal{L} = x^\alpha y^{1-\alpha} - \lambda(x^2 + y^2) + \mu_x x + \mu_y y$. The first-order conditions are

$$0 = \alpha x^{\alpha-1} y^{1-\alpha} - 2\lambda x + \mu_x$$
$$0 = (1-\alpha) x^\alpha y^{-\alpha} - 2\lambda y + \mu_y$$

These cannot be satisifed if $x = 0$ or $y = 0$. Complementary slackness then shows $\mu_x = \mu_y = 0$. We then eliminate λ, finding $(\alpha/(1-\alpha))y^2 = x^2$. Since λ is not zero, $x^2+y^2 = 1$. Substituting, we find $x = \sqrt{\alpha}$ and $y = \sqrt{1-\alpha}$. With one binding constraint, the bordered Hessian is

$$H = \begin{bmatrix} 0 & 2x & 2y \\ 2x & \alpha(\alpha-1)x^{\alpha-2}y^{1-\alpha} - 2\lambda & \alpha(1-\alpha)x^{\alpha-1}y^{-\alpha} \\ 2y & \alpha(1-\alpha)x^{\alpha-1}y^{-\alpha} & -\alpha(1-\alpha)x^{\alpha}y^{-\alpha-1} - 2\lambda \end{bmatrix}.$$

Then $H_3 = 8\alpha(1-\alpha)x^\alpha y^{1-\alpha} + 4\alpha(1-\alpha)x^\alpha y^{1-\alpha} + 4\lambda x^2 - 4\alpha(1-\alpha)x^\alpha y^{1-\alpha} + 4\lambda y^2 = 16x^\alpha y^{1-\alpha} + 4\lambda(x^2+y^2) > 0$. Since there are $n = 2$ variables, and H_3 has the sign of $(-1)^n$, we have a maximum.

3. Utility on $\mathbb{R}^2_+$ is given by $u(x,y) = x + 2y$. Using an appropriate Lagrangian, find all utility maximizing points in the set $B = \{(x,y) : 2x + y \le 6,\ x + y \le 4,\ x + 3y \le 9, \text{ and } x, y \ge 0\}$. Don't forget to check the constraint qualification conditions.

Answer: The derivative of the constraints is

$$dg = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A quick look at the constraints shows that at most two can bind simultaneously. Since any two rows of dg are linearly independent, NDCQ is statisfied. Alternatively, you can use the fact that all the constraints are linear.

The Lagrangian is $\mathcal{L} = x+2y-\lambda_1(2x+y-6)-\lambda_2(x+y-4)-\lambda_3(x+3y-9)+\mu_x x+\mu_y y$. The first order conditions are

$$0 = 1 - 2\lambda_1 - \lambda_2 - \lambda_3 + \mu_x$$
$$0 = 2 - \lambda_1 - \lambda_2 - 3\lambda_3 + \mu_y$$

If $x = 0$, the first two constraints are slack ($y \le 6$ and $y \le 4$), so $\lambda_1 = \lambda_2 = 0$. Then $\lambda_3 > 0$, so $y = 3$. This implies $\lambda_3 = 2/3$ and $\mu_x = -1/3 < 0$, so it is not a solution.

Now we know $x > 0$ and $\mu_x = 0$. Suppose $y = 0$. Then constraints (2) and (3) are slack ($x \le 4$ and $x \le 9$) since constraint (1) tells us $x \le 3$. Now $\lambda_2 = \lambda_3 = 0$. Here $\lambda_1 = 1/2$ and $\mu_x = -3/2 < 0$, so this is not a solution either.

We must have both $x > 0$ and $y > 0$, so $\mu_x = \mu_y = 0$. If constraint (1) binds, constraint (2) implies $x \geq 2$ while constraint 3 say $x \geq 9/5$. Constraint (3) cannot bind, so $\lambda_3 = 0$. This implies $1 = 2\lambda_1 + \lambda_2$ and $2 = \lambda_1 + \lambda_2$, so $\lambda_1 = -1$, which is impossible. This shows that constraint (1) cannot bind.

Now we also have $\lambda_1 = 0$ and the first order conditions reduce to

$$1 = \lambda_2 + \lambda_3$$
$$2 = \lambda_2 + 3\lambda_3.$$

This implies $\lambda_2 = 1/2$ and $\lambda_3 = 1/2$. Both constraints (2) and (3) must bind. Thus $x + y = 4$ and $x + 3y = 9$. The solution is $(x, y) = (3/2, 5/2)$.

4. On $\mathbb{R}^1$, define $f(x) = x^2 - 4x$. Is f quasi-concave? Is f quasi-convex?

Answer: Let $g(x) = f(x) + 4 = x^2 - 4x + 4 = (x - 2)^2$. Since the upper and lower contour sets of f are just translates of the contour sets for g, it will be enough to consider g.

The upper contour set $\{x : g(x) \geq \alpha\}$ is $(-\infty, +\infty)$ if $\alpha \leq 0$, and $(-\infty, 2 - \sqrt{\alpha}] \cup [2 + \sqrt{\alpha}, +\infty)$. This is not convex for $\alpha > 0$, so g and f are not quasi-concave.

The lower contour set $\{x : g(x) \geq \alpha\}$ is empty if $\alpha < 0$, and $[2 - \sqrt{\alpha}, 2 + \sqrt{\alpha}]$ when $\alpha \geq 0$. These intervals are convex, so g and f are quasi-convex.