Mathematical Economics Exam #1, September 30, 2013

1. Consider $S = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1 \text{ and } -1 < y < 1\}$. Determine whether S is a closed set, open set, or neither. Justify your answer.

   Answer: The set S is neither open nor closed. To see it is not closed, consider the sequence $\mathbf{x}_n = (0, 1 - 1/n)^T$. Then $\mathbf{x}_n \to (0, 1) \notin S$. To see it is not open notice that $B_\epsilon(-1, 0)$ contains $(-1-\epsilon/2, 0) \notin S$, so S does not contain any open balls about $(-1, 0)^T$.

2. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ -3 \\ -3 \\ -3 \end{pmatrix}$$

   a) Do $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ span $\mathbb{R}^4$? Explain

   b) Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a linearly independent set of vectors? Explain

   c) Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a basis for $\mathbb{R}^4$? Why or why not?

   Answer:

   a) The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ cannot span $\mathbb{R}^4$ because their span has dimension at most 3.

   b) No, the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are not linearly independent because $3\mathbf{x}_2 - 3\mathbf{x}_1 + \mathbf{x}_3 = \mathbf{0}$.

   c) The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are not a basis for $\mathbb{R}^4$ as they neither span $\mathbb{R}^4$ nor are linearly independent.

3. Demand for good 1 is $e_1 - ap_1 + bp_2$; demand for good 2 is $e_2 + cp_1 - dp_2$; the supply of good i is s_i. Here a, b, c, d, e_i, and s_i are all positive, and $s_i > e_i$.

   a) What system of equations do you get when you set supply equal to demand in both markets?

   b) What criterion must be met in order to solve for p_1 and p_2?

   c) Give additional conditions must be satisfied in order to get positive equilibrium prices p_i?

   Answer:

a) The equations are

$$s_1 = e_1 - ap_1 + bp_2$$
$$s_2 = e_2 + cp_1 - dp_2$$

or

$$s_1 - e_1 = -ap_1 + bp_2$$
$$s_2 - e_2 = cp_1 - dp_2$$

b) This will have a solution if $\det \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} = ad - bc \neq 0$.

c) Using Cramer's Rule (or whatever method you prefer) the solutions are

$$p_1 = -\frac{d(s_1 - e_1) + b(s_2 - e_2)}{ad - bc}$$

and

$$p_2 = -\frac{a(s_2 - e_2) + c(s_1 - e_1)}{ad - bc}$$

Since $[d(s_1 - e_1) + b(s_2 - e_2)] > 0$ and $[a(s_2 - e_2) + c(s_1 - e_1)] > 0$, the solutions will be positive if $ad - bc < 0$.

4. Consider the linear system

$$w - x + 3y - z = 0$$
$$w + 4x - y + z = 3$$
$$3w + 7x + y + z = 6.$$

a) Determine whether this system has solutions.

b) How many basic variables are there? How many free variables?

c) How many solutions are there?

Answer: This system has augmented matrix

$$\begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 4 & -1 & 1 & 3 \\ 3 & 7 & 1 & 1 & 6 \end{pmatrix}.$$

We row-reduce the augmented matrix to determine whether there are solutions:

$$\begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 4 & -1 & 1 & 3 \\ 3 & 7 & 1 & 1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 0 & 5 & -4 & 2 & 3 \\ 0 & 10 & -8 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 0 & 5 & -4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This can be further row-reduced to

$$\begin{pmatrix} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Since both the augmented matrix has the same rank (2) as the unaugmented matrix, the system has a solution.

b) Here there are two basic variables (w and x) and two free variables (y and z)

c) Since there is a solution, and at least one free variable, the system has many solutions.