1. Consider the problem of maximizing \( x + y \) subject to the constraint that \( x^2 + y^2/4 \leq 1 \).

   a) Without calculating it, prove this problem has a solution.

   b) Find the solution.

Answer:

   a) Since the constraint set is compact and \( x + y \) is continuous, the Weierstrass Theorem guarantees there is a solution.

   b) Form the Lagrangian \( \mathcal{L} = x + y - \lambda(x^2 + y^2/4 - 1) \). The first-order conditions are \( 1 = 2\lambda x \) and \( 1 = \lambda y/2 \). Here \( \lambda > 0 \), so \( x^2 + y^2/4 = 1 \) by complementary slackness. Now \( 2x = y/2 \) so \( x = y/4 \) and \( (y/4)^2 + y^2/4 = 1 \). Thus \( y = 4/\sqrt{5} \) and \( x = 1/\sqrt{5} \) is the solution.

2. Let \( u(x, y) = -\exp(-x^2 - y^2) \).

   a) Is \( u \) an increasing function? Explain

   b) Is \( u \) a homogeneous function? Explain

   c) Is \( u \) a homothetic function? Explain

Answer:

   a) Now \( \partial u/\partial x = 2x \exp(-x^2 - y^2) > 0 \) and \( \partial u/\partial y = 2y \exp(-x^2 - y^2) > 0 \) when \( (x, y) \in \mathbb{R}^2 \) (but not all of \( \mathbb{R}^2 \)), so \( u \) is increasing on \( \mathbb{R}^2 \).

   b) We see that \( u(\lambda x, \lambda y) = -\exp(-\lambda^2(x^2 - y^2)) \neq \lambda^a u(x, y) \) for any \( \alpha \), so \( u \) is not homogeneous.

   c) The function \( u \) is homothetic as it is an increasing function \(-e^{-x}\) of a homogeneous of degree 2 function \((x^2 + y^2)^2\).

3. Let \( f(x, y, z) = (x^2 + y^2)z^2 \).

   a) If \((x, y) = (1/2, 1/2)\) and \( f(x, y, z) = 2 \), what values may \( z \) take.

   b) When \((x, y) = (1/2, 1/2)\), can \( z \) be written as a \( C^1 \) function of \((x, y)\) near \((1/2, 1/2)\)? Explain.

   c) If you answered (b) affirmatively, call such a function \( g \) and find \( \text{dg}(1/2, 1/2) \).

Answer:

   a) Set \( 2 = f(1/2, 1/2, z) = z^2/2 \). Then solve to find \( z = \pm 2 \).

   b) We calculate \( \partial f/\partial z = 2z(x^2 + y^2) \). Using \((x, y, z) = (1/2, 1/2, \pm 2)\), we find \( \partial f/\partial z = \pm 2 \). As this is non-zero, we can apply the Implicit Function Theorem about either solution to write \( z \) as a function of \((x, y)\).

   c) The Implicit Function Theorem tells us that \( \text{dg}(1/2, 1/2) = (\partial f/\partial z)^{-1} d_{(x,y)} f \). We already found \( \partial f/\partial z = \pm 2 \). Now \( d_{(x,y)} f(1/2, 1/2, \pm 2) = (4, 4) \) so \( df(1/2, 1/2) = \pm(2, 2) \) depending on which value of \( z \) we picked.

4. Utility on \( \mathbb{R}^2 \) is given by \( u(x, y) = x^\alpha y^{1-\alpha} \) where \( 0 < \alpha < 1 \).

   a) Find all the points that maximize utility over the set \( B = \{(x, y) : 2x + y \leq 9, x + 2y \leq 9, \text{and } x, y \geq 0\} \).

   b) Show that an appropriate constraint qualification condition holds.

Answer:
We form the Lagrangian \( L = x^\alpha y^{1-\alpha} - \lambda_1(2x+y-9) - \lambda_2(x+2y-9) + \mu_x x + \mu_y y \). We note that \( x = 0 \) or \( y = 0 \) yields zero utility (a minimum), so \( x > 0 \) and \( y > 0 \) at the maximum. Complementary slackness yields \( \mu_x = \mu_y = 0 \) and the first order conditions become \( \alpha x^{\alpha-1} y^{1-\alpha} = 2\lambda_1 + \lambda_2 \) and \( (1-\alpha)x^\alpha y^{-\alpha} = \lambda_1 + 2\lambda_2 \). There are 3 cases: only constraint 1 binds, only constraint 2 binds, both 1 and 2 bind.

If only constraint 1 binds, \( \lambda_2 = 0 \). The first order conditions are \( \alpha x^{\alpha-1} y^{1-\alpha} = 2\lambda_1 \) and \( (1-\alpha)x^\alpha y^{-\alpha} = \lambda_1 \). Dividing, we find \((xy)/(1-\alpha)x) = 2\). Using the binding constraint yields \((x,y) = (9\alpha/2, 9(1-\alpha)) \). We now check the other constraint, \( 9\alpha/2 + 18(1-\alpha) \leq 9 \). This only holds if \( \alpha \leq 1/3 \).

If only constraint 2 binds, \( \lambda_1 = 0 \). The first order conditions are \( \alpha x^{\alpha-1} y^{1-\alpha} = \lambda_2 \) and \( (1-\alpha)x^\alpha y^{-\alpha} = 2\lambda_2 \). Dividing, we find \((xy)/(1-\alpha)x) = 1/2 \). Using the binding constraint yields \((x,y) = (9\alpha, 9(1-\alpha)/2) \). We now check the other constraint, \( 18\alpha + 9(1-\alpha)/2 \leq 9 \). This only holds if \( \alpha \leq 1/3 \).

Finally, if both constraints bind, \((x,y) = (3,3) \). In this case, the first order conditions become \( \alpha = 2\lambda_1 + \lambda_2 \) and \( 1-\alpha = \lambda_1 + 2\lambda_2 \). Here \( \lambda_1 \geq 0 \) and \( \lambda_1 + \lambda_2 = 1/3 \). It follows that \( 1/3 \leq \alpha \leq 2/3 \).

Summing up, the solution is
\[
(x,y) = \begin{cases} 
(9\alpha, 9(1-\alpha)/2) & \text{if } 0 < \alpha \leq 1/3 \\
(3,3) & \text{if } 1/3 \leq \alpha \leq 2/3 \\
(9\alpha/2, 9(1-\alpha)) & \text{if } 2/3 \leq \alpha < 1 
\end{cases}
\]

The four constraints have derivative
\[
dg = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

By sketching the constraint set, we see that only 2 constraints can bind at any one time. Since no two pairs of rows are linearly independent, and all rows are non-zero, the rank of \( dg \) will equal the number of binding constraints (1 or 2).