Mathematical Economics Exam #2, November 4, 2013

1. Consider the problem of maximizing $x + y$ subject to the constraint that $x^2 + y^2/4 \leq 1$.

   a) Without calculating it, prove this problem has a solution.

   b) Find the solution.

   Answer:

   a) Since the constraint set is compact and $x + y$ is continuous, the Weierstrass Theorem guarantees there is a solution.

   b) Form the Lagrangian $\mathcal{L} = x + y - \lambda(x^2 + y^2/4 - 1)$. The first-order conditions are $1 = 2\lambda x$ and $1 = \lambda y/2$. Here $\lambda > 0$, so $x^2 + y^2/4 = 1$ by complementary slackness. Now $2x = y/2$ so $x = y/4$ and $(y/4)^2 + y^2/4 = 1$. Thus $y = 4/\sqrt{5}$ and $x = 1/\sqrt{5}$ is the solution.

2. Let $u(x, y) = -\exp(-x^2 - y^2)$.

   a) is u an increasing function? Explain

   b) Is u a homogeneous function? Explain

   c) Is u a homothethic function? Explain

   Answer:

   a) Now $\partial u/\partial x = 2x \exp(-x^2 - y^2) > 0$ and $\partial u/\partial y = 2y \exp(-x^2 - y^2) > 0$ when $(x, y) \in \mathbb{R}^2_+$ (but not all of $\mathbb{R}^2$), so u is increasing on $\mathbb{R}^2_+$.

   b) We see that $u(\lambda x, \lambda y) = -\exp -\lambda^2(x^2 - y^2) \neq \lambda^\alpha u(x, y)$ for any α, so u is not homogeneous.

   c) The function u is homothetic as it is an increasing function $(-e^{-x})$ of a homogeneous of degree 2 function $(x^2 + y^2)$.

3. Let $f(x, y, z) = (x^2 + y^2)z^2$.

   a) If $(x, y) = (1/2, 1/2)$ and $f(x, y, z) = 2$, what values may z take.

   b) When $(x, y) = (1/2, 1/2)$, can z be written as a C^1 function of (x, y) near $(1/2, 1/2)$? Explain.

   c) If you answered (b) affirmatively, can such a function g and find $dg(1/2, 1/2)$.

   Answer:

   a) Set $2f(1/2, 1/2, z) = z^2/2$. Then solve to find $z = \pm 2$.

   b) We calculate $\partial f/\partial z = 2z(x^2 + y^2)$. Using $(x, y, z) = (1/2, 1/2, \pm 2)$, we find $\partial f/\partial z = \pm 2$. As this is non-zero, we can apply the Implicit Function Theorem about either solution to write z as a function of (x, y).

   c) The Implicit Function Theorem tells us that $dg(1/2, 1/2) = (\partial f/\partial z)^{-1} d_{(x,y)} f$. We already found $\partial f/\partial z = \pm 2$. Now $d_{(x,y)} f(1/2, 1/2, \pm 2) = (4, 4)$ so $df(1/2, 1/2) = \pm(2, 2)$ depending on which value of z we picked.

4. Utility on $\mathbb{R}^2_+$ is given by $u(x, y) = x^\alpha y^{1-\alpha}$ where $0 < \alpha < 1$.

   a) Find all the points that maximize utility over the set $B = \{(x, y) : 2x + y \leq 9,\ x + 2y \leq 9, \text{ and } x, y \geq 0\}$.

   b) Show that an appropriate constraint qualification condition holds.

   Answer:

a) We form the Lagrangian $\mathcal{L} = x^\alpha y^{1-\alpha} - \lambda_1(2x+y-9) - \lambda_2(x+2y-9) + \mu_x x + \mu_y y$. We note that $x = 0$ or $y = 0$ yields zero utility (a minimum), so $x > 0$ and $y > 0$ at the maximum. Complementary slackness yields $\mu_x = \mu_y = 0$ and the first order conditions become $\alpha x^{\alpha-1}y^{1-\alpha} = 2\lambda_1 + \lambda_2$ and $(1-\alpha)x^\alpha y^{-\alpha} = \lambda_1 + 2\lambda_2$. There are 3 cases: only constraint 1 binds, only constraint 2 binds, both 1 and 2 bind.

If only constraint 1 binds, $\lambda_2 = 0$. The first order conditions are $\alpha x^{\alpha-1}y^{1-\alpha} = 2\lambda_1$ and $(1-\alpha)x^\alpha y^{-\alpha} = \lambda_1$. Dividing, we find $(\alpha y)/((1-\alpha)x) = 2$. Using the binding constraint yields $(x, y) = \big(9\alpha/2, 9(1-\alpha)\big)$. We now check the other constraint, $9\alpha/2 + 18(1-\alpha) \leq 9$. This only holds if $2/3 \leq \alpha$.

If only constraint 2 binds, $\lambda_1 = 0$. The first-order conditions are $\alpha x^{\alpha-1}y^{1-\alpha} = \lambda_2$ and $(1-\alpha)x^\alpha y^{-\alpha} = 2\lambda_2$. Dividing, we find $(\alpha y)/((1-\alpha)x) = 1/2$. Using the binding constraint yields $(x, y) = \big(9\alpha, 9(1-\alpha)/2\big)$. We now check the other constraint, $18\alpha + 9(1-\alpha)/2 \leq 9$. This only holds if $\alpha \leq 1/3$.

Finally, if both constraints bind, $(x, y) = (3, 3)$. In this case, the first-order conditions become $\alpha = 2\lambda_1 + \lambda_2$ and $1-\alpha = \lambda_1 + 2\lambda_2$. Here $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1/3$. It follows that $1/3 \leq \alpha \leq 2/3$.

Summing up, the solution is

$$(x, y) = \begin{cases} \big(9\alpha, 9(1-\alpha)/2\big) & \text{if } 0 < \alpha \leq 1/3 \\ \big(3, 3\big) & \text{if } 1/3 \leq \alpha \leq 2/3 \\ \big(9\alpha/2, 9(1-\alpha)\big) & \text{if } 2/3 \leq \alpha < 1 \end{cases}$$

b) The four constraints have derivative

$$dg = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By sketching the constraint set, we see that only 2 constraints can bind at any one time. Since no two pairs of rows are linearly independent, and all rows are non-zero, the rank of $d\tilde{g}$ will equal the number of binding constraints (1 or 2).