Mathematical Economics Final, December 13, 2013

1. Let $f(x, y) = x + xy + x^2$. Is f concave or convex on $\mathbb{R}^2_{++}$?

   Answer: We compute the Hessian. $Df = (1 + y + 2x, x)$, so

$$H = D^2 f = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

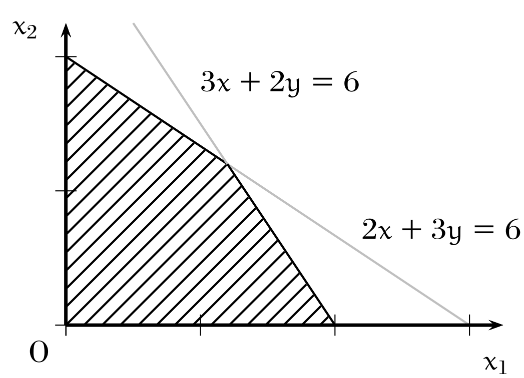
   Here $H_{11} = 2 > 0$ and $\det H = -1 < 0$. Since $H_{11} > 0$, it cannot negative definite, and since $\det H < 0$, it cannot be positive definite. The matrix H violates both sign patterns, and it follows it is indefinite. The function f is neither concave nor convex.

2. A consumer has utility function $u(x, y) = x + y$. The consumer consumes non-negative quantities of both goods, subject to two budget constraints: $3x + 2y \leq 6$ and $2x + 3y \leq 6$. Find (x^*, y^*) that maximizes utility subject to the above four constraints. Be sure to check the constraint qualification.

   Answer: This sort of problem can arise when one of the goods is rationed via ration coupons, and there is a market for ration coupons where the relative price for coupons is different than for goods. We first consider constraint qualification. There are four constraints: $3x + 2y - 6 \leq 0$, $2x + 3y - 6 \leq 0$, $-x \leq 0$, and $-y \leq 0$. The matrix of derivatives of the constraints is:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Constraint qualification will be satisfied as long as at most 2 constraints bind. Examining the constraints shows that at most 2 can bind (see the diagram).



The Lagrangian is $\mathcal{L} = x + y - \lambda(3x + 2y - 6) - \mu(2x + 3y - 6) + \nu_x x + \nu_y y$. The first-order conditions are $1 - 3\lambda - 2\mu + \nu_x = 0$ and $1 - 2\lambda - 3\mu + \nu_y = 0$. There are 4 cases to consider.

1) $x, y > 0$. Complementary slackness implies $\nu_x = \nu_y = 0$. Solving the first-order condition yields $\lambda = \mu = 1/5$. The corresponding constraints both bind by complementary slackness, yielding $3x + 2y = 6$ and $2x + 3y = 6$. The correspoding critical point is $(x, y) = (6/5, 6/5)$.

2) $x = 0$, $y > 0$. Here $\nu_y = 0$ by complementary slackness. The two budget constraints reduce to $2y \le 6$ and $3y \le 6$. The first cannot bind, so its multiplier (λ) is zero. The first-order conditions become $1 + \nu_x = 2\mu$ and $1 = 3\mu$. Since $\mu = 1/3$, $\nu_x < 0$, which is impossible at a maximum.

3) $x > 0$, $y = 0$. Here $\nu_x = 0$ and the budget constraints become $3x \le 6$ and $2x \le 6$. The second cannot bind, so $\mu = 0$. The first-order conditions become $1 = 3\lambda$ and $1 + \nu_y = 2\lambda$. This implies $\nu_y < 0$, which is impossible at a maximum.

4) $x = y = 0$. Here $\lambda = \mu = 0$ by complementary slackness leaving the equations $1 + \nu_x = 0$ and $1 + \nu_y = 0$. Since $\nu_x, \nu_y \ge 0$ at a maximum, this too is impossible.

In sum, there is only one critical point, $(x, y) = (6/5, 6/5)$, so we don't need to worry about second-order conditions.

3. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Find $\exp(tA)$.

Answer: The characteristic equation is $\lambda^2 - 6\lambda + 8 = 0$, yielding eigenvalues $\sigma(A) = \{2, 4\}$. The corresponding eigenvectors are $\mathbf{v}_2 = (1, 1)^T$ and $\mathbf{v}_4 = (1, -1)^T$. Let $P = [\mathbf{v}_2, \mathbf{v}_4]$. We have

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This implies $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = P^{-1}AP$. Then $\sqrt{A} = P\sqrt{D}P^{-1}$. Since

$$\exp(tD) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix}, \quad \exp(tA) = \frac{1}{2}\begin{pmatrix} e^{2t} + e^{4t} & e^{2t} - e^{4t} \\ e^{2t} - e^{4t} & e^{2t} + e^{4t} \end{pmatrix}.$$

4. Consider the difference equation $x_{t+2} - 9x_t = 8$.

a) Find all steady state solutions to the difference equation above.

b) Find the solution to above equation with $x_0 = 0$, $x_1 = 2$.

c) Does the solution in (b) converge to the steady state?

Answer:

a) Setting $x_t = \bar{x}$, we find $-8\bar{x} = 1$, and the only steady state solution is $\bar{x} = -1$.

b) We know that the general solution is $x_t = -1 + z_t$ where z_t satisifes the homogeneous equation $z_{t+2} - 9z_t = 0$. Substituting $z_t = \lambda^t$ we find $\lambda^2 - 9 = 0$, so $\lambda = \pm 3$. This implies the general solution has the form $x_t = -1 + \alpha 3^t + \beta(-3)^t$. The initial conditions imply $0 = -1 + \alpha + \beta$ and $2 = -1\alpha - 3\beta$. Solving, we obtain $\alpha = 1$ and $\beta = 0$, so the solution $x_t = -1 + 3^t$ satisfies the difference equation and the inital conditions.

c) The solution $x_t = -1 + 3^t$ does not converge anywhere.

5. On $\mathbb{R}^2_+$, let $u(x_1, x_2) = \sqrt{x_1} + 4\sqrt{x_2}$. Suppose $\mathbf{p} \gg \mathbf{0}$ and $m > 0$. Maximize u subject to the constraints $x_\ell \geq 0$ and $\mathbf{p} \cdot \mathbf{x} \leq m$ to find the Marshallian demand function $\mathbf{x}(\mathbf{p}, m)$. Don't forget to check constraint qualification and concavity of the objective (or an appropriate second-order condition).

Answer: We consider constraint qualification first. The matrix of derivatives of the constraints is:

$$\begin{bmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

At most two of the constraints can bind. Since any two rows of this matrix are linearly independent, constraint qualification will be satisfied regardless of which constraints bind. We also note that the objective u is strictly concave because its Hessian

$$H = \begin{pmatrix} -\frac{4}{x_1^{3/2}} & 0 \\ 0 & -\frac{1}{x_2^{3/2}} \end{pmatrix}$$

is negative definite ($H_1 < 0$ and $\det H > 0$).

The Lagrangian is $\mathcal{L} = \sqrt{x_1} + \sqrt{x_2} - \lambda(p_1x_1 + p_2x_2 - m) + \mu_1 x_1 + \mu_2 x_2$, yielding first-order conditions $1/2\sqrt{x_1} + \mu_1 = \lambda p_1$ and $2/\sqrt{x_2} + \mu_2 = \lambda p_2$. These cannot be satisfied if $x_1 = 0$ or $x_2 = 0$. Thus $x_1, x_2 > 0$ and $\mu_1 = \mu_2 = 0$ by complemetary slackness.

Eliminating λ, we find $\sqrt{x_2}/4\sqrt{x_1} = p_1/p_2$. Squaring and rearranging yields $x_2 = (16p_1^2/p_2^2)x_1$. We substitute in the budget constraint to find

$$p_1 x_1 + \frac{16p_1^2}{p_2} x_1 = m.$$

Thus

$$x_1 = \left(\frac{p_2}{p_1}\right) \frac{m}{p_2 + 16p_1} \text{ and } x_2 = 16\left(\frac{p_1}{p_2}\right) \frac{m}{p_2 + 16p_1}.$$