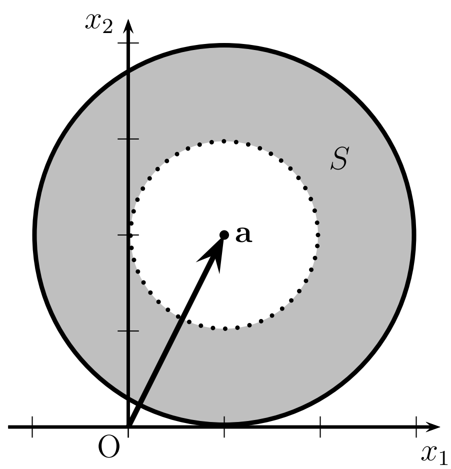
Mathematical Economics Exam #1, September 30, 2014

1. Let $\mathbf{a} = (1, 2)$ and consider $S = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{a}\| \le 2 \text{ and } \|\mathbf{x} - \mathbf{a}\| > 1\}$. Determine whether S is a closed set, open set, or neither. Justify your answer.

   Answer: The set S is neither open nor closed. To see it is not closed, consider the sequence $\mathbf{x}^n = (2 + 1/n, 2)$. Then $\|\mathbf{x}^n - \mathbf{a}\| = \|(1 + 1/n, 0)\| = 1 + 1/n$, so $\mathbf{x}^n \in S$ for $n > 2$. But $\mathbf{x}^n \to (2, 2)$ and $\|(2, 2) - \mathbf{a}\| = 1$, so $(2, 2) \notin S$. This shows that S is not closed. Now consider the point $(3, 2)$. Clearly, $(3, 2) \in S$. Now let $\epsilon > 0$. Then $(3 + \epsilon/2, 2) \in B_\epsilon(3, 2)$, but $\|(3 + \epsilon/2, 2) - \mathbf{a}\| = 2 + \epsilon/2 > 2$, so $(3 + \epsilon/2, 2) \notin S$. This shows that no epsilon ball about $(3, 2)$ can be contained in S, so S is not open.



2. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{x}_3 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ -2 \end{pmatrix}$$

   a) Find the the dimension of the span of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

   b) Is $(1, 1, 1, 1)$ in the span of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$?

   c) Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ a basis for $\mathbb{R}^4$? Why or why not?

   Answer:

a) We form the matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 2 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

We then row-reduce A, obtaining

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which has rank 3. The dimension of the span is $\operatorname{rank} A = 3$.

b) Yes. $(1,1,1,1) = \mathbf{x}_2 + \frac{1}{2}\mathbf{x}_3$.

c) The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are not a basis for $\mathbb{R}^4$ as they do not span $\mathbb{R}^4$ (the span has dimension 3, not 4).

3. Consider the sequence $\{x_n\}_{n=1}^\infty = \{1 + (-1)^n\}_{n=1}^\infty$. Answer (a) and either (b) or (c).

a) Does the sequence $\{x_n\}_{n=1}^\infty$ converge?

b) If you answered yes to (a), what is the limit? Prove it.

c) If you answered no to (a), is there a convergent subsequence of $\{x_n\}_{n=1}^\infty$? What is its limit? Prove it.

Answer:

a) No. The sequence is $\{0, 2, 0, 2, \ldots\}$. Suppose there is a limite x. Choose N large enough that $|x_n - x| < 1$ for $n \geq N$. Then $|x_n - x_{n+1}| < 2$ for $n \geq N$. But $|x_n - x_{n+1}| = 2$. Since supposing there is a limit leads to a contradiction, there is no limit.

b) N/A.

c) Two convergent subsequences are $x_{n_i} = x_{2i} = 2$ and and $x_{n_i} = x_{2i-1} = 0$. The first converges to 2 as $|x_{n_i} - 2| = 0 < \epsilon$ for all $\epsilon > 0$ and $n \geq 1$. The second converges to zero as $|x_{n_i} - 0| = 0 < \epsilon$ for all $\epsilon > 0$ and $n \geq 1$.

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 10 \\ 0 & 0 & 14 \end{pmatrix}.$$

a) For what values of λ is $A - \lambda I$ **not** invertible? Denote the set of non-invertible values by Λ.

b) For each $\lambda \in \Lambda$, find a non-zero vector $\mathbf{x}$ such that $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

c) Do the three vectors you found in part (b) span $\mathbb{R}^3$? Do they form a basis for $\mathbb{R}^3$?

Answer:

a) We take the determinant $\det(A - \lambda I) = (1 - \lambda)(5 - \lambda)(14 - \lambda)$ The matrix is not invertible if and only if the determinant is zero. This happens only when $\lambda \in \Lambda = \{1, 5, 14\}$.

b) We find non-zero vectors solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$. When $\lambda = 1$, $\mathbf{x} = (1, 0, 0)$ is a solution. When $\lambda = 5$, $\mathbf{x} = (3, 1, 0)$ is a solution. When $\lambda = 14$, $\mathbf{x} = (\frac{47}{13}, 10, 9)$ is a solution.

c) The matrix

$$\begin{pmatrix} 1 & 3 & \frac{47}{13} \\ 0 & 1 & 10 \\ 0 & 0 & 9 \end{pmatrix}$$

has determinant 9. As this is non-zero, the rank of the matrix of vectors is 3. Since the rank is the number of rows, the vectors span $\mathbb{R}^3$ and since the rank is also the number of columns, the vector are linearly independent. Thus they form a basis for $\mathbb{R}^3$.