1. Consider the function \( f(t) = \begin{bmatrix} t^2 \\ -t \end{bmatrix} \).

   a) Compute the tangent vector of \( f \) at any \( t \).

   b) Give an equation for the tangent line at the point \( \begin{bmatrix} 4 \\ -2 \end{bmatrix} \).

   **Answer:**

   a) The tangent vector is given by the derivative \( \frac{df}{dt} = \begin{bmatrix} 2t \\ -1 \end{bmatrix} \).

   b) The point is \( f(2) \), so the tangent vector is \( \begin{bmatrix} 4 \\ -1 \end{bmatrix} \). There are several ways to write the tangent line. One is that

   \[
   L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\}.
   \]

   It can also be written by eliminating \( t \) from the equations. For that, \( y = -2 - t \), so \( t = -(2 + y) \). Substituting in \( x = 4 + 4t \) yields \( -4 = x + 4y \).

2. Let \( f: \mathbb{R}^2_+ \to \mathbb{R}_+ \) be defined by \( f(x, y) = x^{2/3}y^{1/3} \).

   a) Find the level curves of \( f \).

   b) Use the Implicit Function Theorem to show that \( y(x) \) defined by \( f(x, y(x)) = q \) for \( q > 0 \) defines a \( C^1 \) function \( y \).

   c) Using \( y \) as in part (b), compute \( \frac{dy}{dx} \).

   d) What happens if \( q = 0 \) in part (b). In particular, can you still compute \( \frac{dy}{dx} \)?

   **Answer:**

   a) The level curves are \( \{ (x, y) \in \mathbb{R}^2_+ : x^{2/3}y^{1/3} = q \} \) for \( q \geq 0 \).

   b) We examine whether \( \frac{\partial f}{\partial y} = (1/3)x^{2/3}y^{-2/3} \neq 0 \) for \( x, y \neq 0 \). This condition is satisfied since \( x^{2/3}y^{1/3} = q > 0 \). Because the partial derivative is non-zero, the Implicit Function Theorem tells us \( y(x) \) is a \( C^1 \) function.

   c) Using the Implicit Function Theorem, we find

   \[
   \frac{dy}{dx} = -\left( \frac{1}{3} \frac{x^{2/3}}{y^{2/3}} \right)^{-1} \times \left( \frac{2}{3} \frac{y^{1/3}}{x^{1/3}} \right) = -\frac{2y}{x}.
   \]
d) If $q = 0$, the level curve consists of both the non-negative $x$-axis and non-negative $y$-axis. When $x > 0$, $y(x) = 0$, and when $x = 0$, $y(x)$ can be any non-negative number. When $x > 0$, $dy/dx = 0$, and when $x = 0$, $dy/dx$ is undefined.

3. Consider the problem of maximizing $3x + 4y$ subject to the constraint that $x^2 + y \leq 5$, $x \geq 0$ and $y \geq 0$.

   a) Without calculating it, prove this problem has a solution.

   b) Find the solution. Don’t forget to check constraint qualification.

Answer:

   a) Since the constraint set is compact (closed and bounded) and $3x + 4y$ is continuous, the Weierstrass Theorem guarantees there is a solution.

   b) It is obvious that all three constraints cannot simultaneously bind. We consider the matrix

   $\begin{bmatrix}
   2x & 1 \\
   -1 & 0 \\
   0 & -1
   \end{bmatrix}$.

   All rows are non-zero, and any two rows are linearly independent provided $x > 0$. When $x = 0$, we must include the second row when either other row will give us a linearly independent pair. This implies that constraint qualification holds.

   Now form the Lagrangian $\mathcal{L} = 3x + 4y - \lambda(x^2 + y - 5) + \mu_x x + \mu_y y$. The first-order conditions are

   $3 = 2\lambda x - \mu_x$, and

   $4 = \lambda - \mu_y$.  \hspace{1cm} (1)

   Here $\lambda \geq 4 + \mu_y \geq 4 > 0$, so $x^2 + y = 5$ by complementary slackness. There are three cases to consider.

   If $x = 0$, then (1) becomes $3 = -\mu_x \leq 0$, which is impossible.

   If $y = 0$, then $x = \sqrt{5}$ and $\mu_x = 0$ by complementary slackness. It follows that $\lambda = 3/2\sqrt{5}$ by (1), which contradicts (2).

   The only possibility left is $x > 0$ and $y > 0$. Then $\mu_x = \mu_y = 0$ by complementary slackness. This implies $3 = 2\lambda x$ and $\lambda = 4$. Thus $x = 3/8$, so
\[ y = 5 - (3/8)^2 = 311/64. \] As the only remaining option, (3/8, 311/64) must be the maximum.

4. Consider the quadratic form \( Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + 3x_1x_3 - x_3^2 \) with constraint \( x_2 + x_3 = 0 \). Does this problem have a maximum, minimum, or saddlepoint at (0, 0, 0)? Explain why.

**Answer:** We form the bordered Hessian

\[
H = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 3/2 \\
1 & 1 & 1 & 0 \\
1 & 3/2 & 0 & -1 \\
\end{bmatrix}
\]

There are three variables \((n = 3)\) and one linear constraint \((m = 1)\), so we look at the last \(n - m = 2\) leading principal minors. They are \(H_3 = -1\) and \(H_4 = +1/4\). Since \(H_4(-1)^n = H_4(-1)^m = -1/4 < 0\), the quadratic form fails both the tests for positive definiteness and negative definiteness on the constraint set. As \(H_4\) is non-zero, we may conclude that \(H\) is indefinite on the constraint set and that \((0, 0, 0)\) is a saddlepoint.

Alternatively, consider the points \(x_1 = \epsilon(1, 2, -2)\) and \(x_2 = \epsilon(1, -2, 2)\). Both satisfy the constraint and \(Q(x_1) = -\epsilon^2\) and \(Q(x_2) = 3\epsilon^2\). This shows that \(Q\) takes both positive and negative values in any neighborhood of 0, so 0 is neither a constrained local max, nor constrained local min.