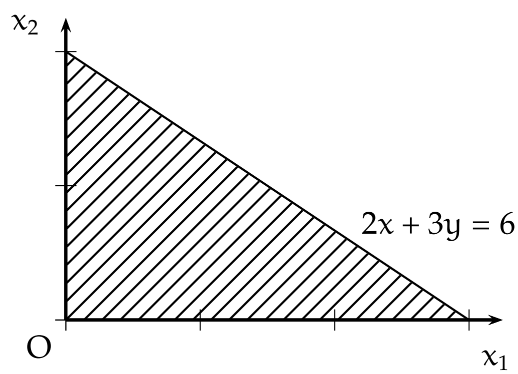
Mathematical Economics Final, December 9, 2014

1. A consumer has the unusual utility function $u(x, y) = x^2 + y^2$. The consumer consumes non-negative quantities of both goods, subject to the budget constraint: $2x + 3y \le 6$. Find (x^*, y^*) that maximizes utility subject to the above three constraints. Be sure to check the constraint qualification.

   Answer: There are three constraints: $2x+3y-6 \le 0$, , $-x \le 0$, and $-y \le 0$. The matrix of derivatives of the constraints is:

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Constraint qualification will be satisfied as long as at most 2 constraints bind. Examining the constraints shows that at most 2 can bind (see the diagram).



The Lagrangian is $\mathcal{L} = x^2 + y^2 - \lambda(2x + 3y - 6) + \mu_x x + \mu_y y$. The first-order conditions are $2x - 2\lambda + \mu_x = 0$ and $2y - 3\lambda + \mu_y = 0$. Since utility is increasing, there are 3 cases to consider.

1) $x, y > 0$. Complementary slackness implies $\mu_x = \mu_y = 0$. According to the first-order conditions $2x = 2\lambda$ and $2y = 3\lambda$, so $\lambda = x = 2y/3 > 0$. Complementary slackness implies $2x+3y = 6$. We conclude $x = \lambda = 12/13$ and $y = 18/13$. In this case $u = 468/169$.

2) $x = 0$, $y > 0$. Here $\mu_y = 0$, so $(3/2)\lambda = y > 0$. Complementary slackness implies $2x + 3y = 6$, so $y = 2$ and $\lambda = 4/3$. Note $\mu_x = 2\lambda = 8/3$. We find $u = 4$.

3) $x > 0$, $y = 0$. Here $\mu_x = 0$ and $\lambda = x > 0$. By the budget constraint, $x = 3$. Note $\mu_y = 3\lambda = 9$. Here $u = 9$.

Thus the maximium is at $(x, y) = (3, 0)$ where $u = 9$.

2. Let $A = \begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix}$. Find $\exp(tA)$.

Answer: The characteristic equation is $\lambda^2 - 6\lambda = 0$, yielding eigenvalues $\sigma(A) = \{0, 6\}$. The corresponding eigenvectors are $\boldsymbol{v}_0 = (1, 3)^T$ and $\boldsymbol{v}_6 = (1, -3)^T$. Let $P = [\boldsymbol{v}_0, \boldsymbol{v}_6]$. We have

$$P = \begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}.$$

This implies $D = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} = P^{-1}AP$. Then

$$\exp(tA) = P\exp(tD)P^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{6t} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3+3e^{6t} & 1-e^{6t} \\ 9-9e^{6t} & 3+3e^{6t} \end{pmatrix}.$$

3. Consider the differential equation $\ddot{y} - \dot{y} - 2y = 0$.
   a) Find the general solution to the differential equation.
   b) Find the solution to above equation with $y(0) = 0$, $\dot{y}(0) = 2$.
   c) Find all solutions to the differential equation that obey $\lim_{t\to+\infty} y(t) = 0$.

Answer:

a) We substitute $y = e^{\lambda t}$ to find the characteristic equation, $\lambda^2 - \lambda - 2 = 0$. This has solutions $\lambda \in \{-1, 2\}$. The general solution is then $y(t) = c_1 e^{-t} + c_2 e^{2t}$.

b) Using the general solution $y(t) = c_1 e^{-t} + c_2 e^{2t}$ we find $0 = y(0) = c_1 + c_2$ and $2 = \dot{y}(0) = -c_1 + 2c_2$. This implies $c_1 = -2/3$ and $c_2 = 2/3$. The solution is $y(t) = -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t}$.

c) $\lim_{t\to+\infty} y(t)$ will not exist if $c_2 \neq 0$. If $c_2 = 0$, $\lim_{t\to+\infty} y(t) = c_1 e^{-\infty} = 0$. It follows that the solutions of the form $y(t) = c_1 e^{-t}$ are the required solutions.

4. Let $f(x, y) = 2x + 3y - x^2 - 3xy - y^2$. Is f concave or convex on $\mathbb{R}^2_{++}$?

Answer: We compute the Hessian. $Df = (2 - 2x - 3y, 3 - 3x - 2y)'$, so

$$H = D^2 f = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}.$$

Here $H_{11} = -4 < 0$ and $\det H = -5 < 0$. Since $H_{11} < 0$, it cannot be positive definite, and since $\det H < 0$, it cannot be negative definite. The matrix H violates both sign patterns, and it follows it is indefinite. The function f is neither concave nor convex.

5. On $\mathbb{R}^2_+$, let $u(x_1, x_2) = \sqrt{x_1} + 4\sqrt{x_2}$. Suppose $\mathbf{p} \gg \mathbf{0}$ and $\bar{u} > 0$. Minimize expenditure $\mathbf{p}\cdot\mathbf{x}$ subject to the constraint $u(\mathbf{x}) \geq \bar{u}$ to find the Hicksian demand function $\mathbf{h}(\mathbf{p}, \bar{u})$ and expenditure function $e(\mathbf{p}, \bar{u})$. Don't forget to check constraint qualification.

   Answer: Since square roots only make sense for non-negative numbers, we proceed by imposing the additional constraints $x_1 \geq 0$ and $x_2 \geq 0$.

   We consider constraint qualification first. The matrix of derivatives of the constraints is:

$$\begin{bmatrix} \frac{1}{2\sqrt{x_1}} & \frac{2}{\sqrt{x_2}} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   At most two of the constraints can bind. Since any two rows of this matrix are linearly independent, constraint qualification will be satisfied regardless of which constraints bind.

   The Lagrangian for this minimization problem is $\mathcal{L} = \mathbf{p}{\cdot}\mathbf{x} - \lambda(\sqrt{x_1} + \sqrt{x_2} - \bar{u}) - \mu_1 x_1 - \mu_2 x_2$. This yields first-order conditions $p_1 = \lambda/2\sqrt{x_1} + \mu_1$ and $p_2 = 2\lambda/\sqrt{x_2} + \mu_2$. These cannot be satisfied if $x_1 = 0$ or $x_2 = 0$. Thus $x_1, x_2 > 0$ and $\mu_1 = \mu_2 = 0$ by complemetary slackness.

   The above implies $\lambda > 0$, so $\sqrt{x_1} + 4\sqrt{x_2} = \bar{u}$ by complementary slackness.

   Now $\sqrt{x_1} = \lambda/2p_1$ and $\sqrt{x_2} = 2\lambda/p_2$, so $\bar{u} = \sqrt{x_1} + 4\sqrt{x_2} = \lambda(1/p_1 + 16/p_2)$ and $\lambda = 2p_1p_2\bar{u}/(16p_1 + p_2)$. It follows that the Hicksian demands are $h_1 = x_1 = p_2^2\bar{u}^2/(16p_1 + p_2)^2$ and $h_2 = x_2 = 16p_1^2\bar{u}^2/(16p_1 + p_2)^2$. Then the expenditure function is $p_1h_1 + p_2h_2 = p_1p_2\bar{u}^2/(16p_1 + p_2)$.