Mathematical Economics Final, December 8, 2015

1. Consider the differential equation $\ddot{y} - 2\dot{y} + y = 0$. Find the general solution. Then find the solution that obeys $y(0) = 1$, $\dot{y}(0) = 2$.

   Answer: The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. The only solution is $\lambda = 1$. Since $\lambda = 1$ is repeated, the general solution is $y(t) = \alpha e^t + \beta t e^t$.

   Now $1 = y(0) = \alpha$ and $2 = \dot{y}(0) = \alpha + \beta$ so $\beta = 1$. The solution obeying $y(0) = 1$ and $\dot{y}(0) = 2$ is $y(t) = e^t + te^t$.

2. Minimize the function $u(x, y) = x^2 + y^2$ subject to the constraints $x + y^2 = 3$. Don't forget to check the second-order conditions.

   Answer: The Hessian of u is $H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is positive definite. It follows that any solution to the first-order conditions will be a local minimum.

   The Lagrangian is $\mathcal{L} = x^2 + y^2 - \lambda(x + y^2 - 3)$. The first-order conditions are $2x = \lambda$ and $2y = 2\lambda y$. If $y = 0$, the constraint yields $x = 3$. In that case $\lambda = 6$ and $u(3, 0) = 9$. If $y \neq 0$, $\lambda = 1$ which implies $x = 1/2$. The constraint then tells us that $y = \pm\sqrt{5/2}$ and $u(1/2, \pm\sqrt{5/2}) = 1/4 + 5/2 = 11/4$. The global minima are at $(1/2, \pm\sqrt{5/2})$.

   Alternatively, we can substitute $y^2 = 3 - x$ to obtain $f(x) = x^2 + 3 - x$. Then $f' = 2x - 1$. The second derivative is positive, so $x = 1/2$ is a global minimum. When $x = 1/2$, $y = \pm\sqrt{5/2}$, and the global minima are at $(1/2, \pm\sqrt{5/2})$.

3. Consider the differential system

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x}.$$

   a) Find and solve the characteristic equation.

   b) Is $(0, 0)$ an asymptotically stable steady state? Explain.

   Answer:

   a) The characteristic polynomial is $p(\lambda) = \det(A - \lambda I) = \lambda^+ 2\lambda + 1 + 1 = \lambda^2 + 2\lambda + 2$. Setting $p(\lambda) = 0$, we find $\lambda = -1 \pm i$.

   b) Since both roots have negative real parts, the steady state $(0, 0)$ is asymptotically stable.

4. Consider the problem of maximizing the utility function $u(x, y) = x^4 + y^2$ subject to the budget constraint $x + 2y \leq 10$ and the non-negativity constraints $x \geq 0$, $y \geq 0$.

   a) Does this problem have a solution? Explain?

b) If the problem has a solution, use the Kuhn-Tucker theorem to find it.

Answer:

a) Yes, it has a solution. We are trying to maximize a continuous function over a compact budget set (we showed in class that such sets are compact). The Weierstrass Theorem tells us there is a solution.

b) Notice that the Hessian of u is positive definite. That means that any interior critical point will be a local minimum, not maximum. We need only consider the boundary cases.

There are three constraints. The derivative of the constraints is

$$d\mathbf{g} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As this always has rank two, constraint qualification (NDCQ) is satisfied.

The Lagrangian is $\mathcal{L} = x^4 + y^2 - \lambda(x + 2y - 10) + \mu_x x + \mu_y y$. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = 4x^3 - \lambda + \mu_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y - 2\lambda + \mu_y = 0.$$

Case I: $x, y > 0$. This is the local minimum mentioned above.

Case II: $x = 0$, $y > 0$. Here $\lambda = y > 0$, so the budget constraint binds and $y = 5$. Setting $\mu_x = \lambda$ satisifies the first-order conditions. Note that $u(0, 5) = 25$.

Case III: $x > 0$, $y = 0$. Here $\lambda = 4x^3 > 0$ so the budget constraint binds and $x = 10$. It follows that $\lambda = 4000$. Setting $\mu_y = 8000$ satisifies the first-order conditions. Note that $u(10, 0) = 10,000$.

The maximum is at $(10, 0)$.

5. Is the function $f(x, y, z) = (x^2 + 2xy + zy + z^2)/(x^3 + xyz + 5yz)$ homothetic? Explain.

Answer: No, it is not homothetic. The numerator is homogeneous of degree 2, but the denonominator is not homogeneous. We find

$$f(\lambda x, \lambda y, \lambda z) = \lambda^{-1} \frac{(x^2 + 2xy + zy + z^2)}{(x^3 + xyz + 5yz/\lambda)}.$$

The presence of λ in the denominator indicates that f is not homothetic.

To make it perfectly clear. $f(1,0,0) = 1 = f(0,1,4)$, but if we double everything, $f(2,0,0) = 1/2 \neq 1 = f(0,2,8)$