Mathematical Economics Exam #1, September 29, 2015

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}.$$

   a) What is dimension of the range of A?

   b) Find $\ker A = \{x : Ax = 0\}$.

   c) What is dimension of $\ker A$?

   Answer: We first row reduce A before proceeding.

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -4 & -1 & -3 \\ 0 & -8 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   a) From the row reduction, we see that the rank of A is three, indicating that $Ax = b$ can always be solved. Thus all of $\mathbb{R}^3$ is the range, and $\dim \operatorname{ran} A = 3$.

   b) We can read the solutions to $Ax = 0$ off the row-reduced form of A. We must have $x_4 = 0$, $x_1 = x_2 = -x_3/4$.

   c) Since there is one free variable, the dimension of $\ker A$ is 1.

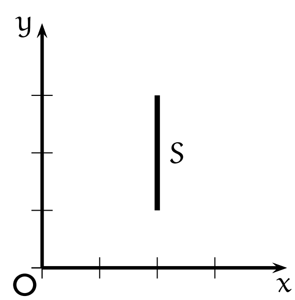
2. Let $S = \{(x, y) : x = 2, 1 \le y \le 3\} \subset \mathbb{R}^2$.

   a) Is S an open set in $\mathbb{R}^2$? If not, find a point x so that $B_\varepsilon(x) \not\subset S$ for any $\varepsilon > 0$.

   b) Is S a closed set in $\mathbb{R}^2$? If not, find a sequence in S that converges to point outside S.

   c) Is S a compact set?

   Answer: The set S is illusrated in the figure.



a) No. The point $x = (2, 2) \in S$, but $(2 + \varepsilon/2, 2) \in B_\varepsilon(x)$ for all $\varepsilon > 0$ and $(2 + \varepsilon/2, 2) \notin S$. Thus S does not contain an open ball around $x \in S$ and so cannot be open.

b) Yes. If $(x_n, y_n) \in S$ converges, $x_n = 2$ and $1 \le y_n \le 3$. Then $\lim x_n = 2$ and $1 \le \lim y_n \le 3$ by Theorem 12.4, so the limit is in S. Thus S contains all its limit points and so is closed.

c) Yes. Compact sets in $\mathbb{R}^2$ are closed and bounded. As seen in part (b), this set is closed. It is also bounded as $\|x\| \le \sqrt{2^2 + 3^2} = \sqrt{13}$. Thus it is compact.

3. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 1 & 5 \end{pmatrix}.$$

a) Use the determinant to find λ such that $A - \lambda I$ is **not** invertible? Denote the set of non-invertible values by Λ.

b) For each $\lambda \in \Lambda$, find a non-zero vector x such that $(A - \lambda I)x = 0$.

c) Do the three vectors you found in part (b) span $\mathbb{R}^3$? Do they form a basis for $\mathbb{R}^3$?

Answer:

a) We take the determinant $\det(A - \lambda I) = (1-\lambda)[(1-\lambda)(5-\lambda)+3] = (1-\lambda)(\lambda^2 - 6\lambda + 8) = (1 - \lambda)(2 - \lambda)(4 - \lambda)$. The matrix is not invertible if and only if the determinant is zero. This happens only when $\lambda \in \Lambda = \{1, 2, 4\}$.

b) We find non-zero vectors solving $(A - \lambda I)x = 0$. When $\lambda = 1$, $x = (1, 0, 0)^T$ is a solution. When $\lambda = 2$, $x = (9, -3, 1)^T$ is a solution. When $\lambda = 4$, $x = (5, -3, 3)^T$ is a solution.

c) The matrix

$$\begin{pmatrix} 1 & 9 & 5 \\ 0 & -3 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

has determinant -6. As this is non-zero, the rank of the matrix of vectors is 3. Since the rank is the number of rows, the vectors span $\mathbb{R}^3$ and since the rank is also the number of columns, the vector are linearly independent. Thus they form a basis for $\mathbb{R}^3$.

4. In $\mathbb{R}^N$, suppose that x is perpendicular to $\{w_1, w_2, \dots, w_n\}$. Show that x is perpendicular to any linear combination of $\{w_1, w_2, \dots, w_n\}$.

Answer: The vector x is perpedicular to each of the w_i, so $0 = w_1 = \cdots w_n$. Consider a linear combination of $z = \sum_{i=i}^n a_i w_i$. Then $x \cdot z = \sum_{i=1}^n a_i (x \cdot w_i)$. Since each $x \cdot w_i = 0$,

$x \cdot z = 0$, proving that any linear combination of the w_i is perpedicular to x.