Mathematical Economics Exam #2, November 3, 2015

1. Consider the problem of maximizing $u(x, y) = x - e^{-y}$ subject to the constraints $x, y \geq 0$ and $x + y \leq 10$.

   a) Is constraint qualification satisfied?

   Answer: The derivative of the constraints is:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   At most two of the constraints can bind. Since all rows are non-zero and any two rows are independent, the rank will eual the number of binding constraints. NDCQ is satisfied.

   b) Find the solution to the maximization problem.

   Answer: The Lagrangian is $\mathcal{L} = x - e^{-y} - \lambda_0(x + y - 10) + \lambda_1 x + \lambda_2 y$. The first-order conditions are:

$$0 = 1 - \lambda_0 + \lambda_1 \tag{1}$$

$$0 = e^{-y} - \lambda_0 + \lambda_2 \tag{2}$$

   Equation (1) tells us $\lambda_0 = 1 + \lambda_1 \geq 1$. It follows that $x + y = 10$ by complementary slackness.

   There are now three cases to consider: 1) $x = 0$, $y = 10$; 2) $x = 10$, $y = 0$; 3) $x, y > 0$.

   In case (1), $\lambda_2 = 0$ by complementary slackness, so $\lambda_0 = e^{-10} < 1$. This violates equation (1). There is no solution here.

   In case (2), $\lambda_1 = 0$ by complementary slackness, so $\lambda_0 = 1$. Using equation (2), we find $\lambda_2 = 0$. This is a possible solution.

   In case (3), $\lambda_1 = \lambda_2 = 0$ by complementary slackness. It follows that $\lambda_0 = 1$. Then equation (2) becomes $e^{-y} = 1$, so $y = 0$, contradicting $y > 0$ in this case.

   It follows that the only solution is $(x, y) = (10, 0)$ with $u(10, 0) = 9$.

2. Let $f(x, y, z) = x^2 + 5y + z^3 - 10$.

   a) Find an (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

Answer: The point $(2, 1, 1)$ works. Other solutions are possible, such as $(1, 2, -1)$ (where the IFT can be used) and $(0, 2, 0)$ (where the IFT fails).

b) Can x be expressed as a function $g(y, z)$ in some neighborhood of (x_0, y_0, z_0)?

Answer: Since $\partial f/\partial x = 2x$, $(\partial f/\partial x)(2, 1, 1) = 4$ The Implicit Function Theorem yields such a function g. Alternatively, note that $g(y, z) = +(10 - 3y - z^3)^{1/2}$ works.

c) Compute dg.

Answer: By the Implicit Function Theorem,

$$dg = -\frac{1}{2x}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = -\frac{1}{4}(5, 3z^2).$$

At $(2, 1, 1)$, this has the value $(-5/4, -3/4)$.

3. Let $f(x, y, z) = x^2 + xy - 2y^2 + z^2 + 2yz$. Find all critical points of f and classify them (local max, local min, saddlepoint, other).

Answer: The first-order conditions are

$$\begin{aligned} 0 &= 2x + y \\ 0 &= x - 4y + 2z \\ 0 &= 2z + 2y. \end{aligned}$$

We can use the last equation to eliminate $z = -y$. The equations then become

$$\begin{aligned} 0 &= 2x + y \\ 0 &= x - 6y. \end{aligned}$$

The only critical point is $(x, y, z) = (0, 0, 0)$. We now consider the Hessian

$$H = d^2 f = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -4 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

The first two leading principal minors are $H_1 = 2$, $H_2 = -9$. $H_2 < 0$ is not consistent with either H being positive definite or negative definite. (Although we didn't need it, $H_3 = -26$.) We conclude H is indefinite and that we have a saddlepoint. In fact $f(\varepsilon, 0, 0) = \varepsilon^2 > 0$ and $f(0, \varepsilon, 0) = -2\varepsilon^2 < 0$, showing that $(0, 0, 0)$ is neither a local maximum nor local minimum.

4. Consider the quadratic form $Q(x, y, z) = x^2 + 4xy + 2y^2 + 6yz - z^2$ with constaint $x + y = 0$.

   a) Find a symmetric matrix that defines this quadratic form.

   Answer: The matrix that defines the quadratic form is

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 3 & -1 \end{pmatrix}.$$

   b) Does the quadratic form have a constrained maximum, minimum, or saddlepoint at $(0, 0, 0)$?

   Answer: We form the bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 0 & 3 & -1 \end{pmatrix}.$$

   There are 3 variables ($n = 3$) and one linear constraint ($m = 1$), so we look at the last $n - m = 2$ leading principal minors. They are $H_3 = +1$ and $H_4 = 8$. Since $(-1)^m H_4 = (-1)^n H_4 = -8 < 0$, the quadratic form fails both the tests for positive definiteness and negative definiteness on the constraint set. As H_4 is non-zero, we may conclude that H is indefinite on the contstraint set and that $(0, 0, 0)$ is a saddlepoint.