

Mathematical Economics Final, December 6, 2016

1. Consider the differential equation $\ddot{y} + 9y = 0$. Find the general solution. Then find the solution that obeys $y(0) = 1$, $\dot{y}(0) = 2$.

Answer: The characteristic equation is $\lambda^2 + 9 = 0$. The solutions are $\lambda = \pm 3i$. In such a case, the general real-valued solution is $y(t) = \alpha \cos 3t + \beta \sin 3t$.

Now $1 = y(0) = \alpha$ and $2 = \dot{y}(0) = 3\beta$ so $\beta = 2/3$. The solution obeying $y(0) = 1$ and $\dot{y}(0) = 2$ is $y(t) = \cos 3t + (3/2) \sin 3t$.

2. Minimize the function $u(x, y) = (x - 1)^2 + (y - 2)^2$ subject to the constraint $x + 3y \leq 3$. Don't forget to check the second-order conditions and constraint qualification.

Answer: Since the constraint is linear, constraint qualification is satisfied.

The Hessian of u is $H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is positive definite. It follows that any solution to the first-order conditions will be a local minimum.

The Lagrangian is $\mathcal{L} = (x - 1)^2 + (y - 2)^2 + \lambda(x + 3y - 3)$. Notice that since this is a minimization problem, we write the constraint as $-x - 3y \geq -3$ when forming the Lagrangian. The first-order conditions are:

$$2x - 2 = -\lambda$$

$$2y - 4 = -3\lambda.$$

If $\lambda = 0$, then $x = 1$ and $y = 2$, which violates the constraint $x + 3y \leq 3$. It follows that $\lambda > 0$ and $x + 3y = 3$ by complementary slackness. But if $\lambda > 0$, we can eliminate λ from the first-order conditions to obtain $2x - 2 = (2/3)y - (4/3)$. Now $x = 3 - 3y$ by complementary slackness and we can substitute to obtain $6 - 6y - 2 = (2/3)y - (4/3)$, or $18 - 18y - 6 = 2y - 4$. Then $y = 4/5$ and $x = 3/5$.

3. Is the function $f(x, y) = x^2 + y^2$ quasiconcave and/or quasiconvex on \mathbb{R}_{++}^2 ? Explain.

Answer: It is convex (positive definite Hessian), and so is quasiconvex.

It is not quasiconcave. We calculate the appropriate bordered Hessian

$$H = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2 & 0 \\ 2y & 0 & 2 \end{pmatrix}.$$

This has determinant $-8x^2 - 8y^2 < 0$ on \mathbb{R}_{++}^2 , showing that the function is not quasiconcave (but again showing it is quasiconvex).

4. Consider the difference equation

$$\mathbf{x}_{n+1} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \mathbf{x}_n.$$

- a) Find the eigenvalues of the system.
- b) Find eigenvectors corresponding to the eigenvalues.
- c) Do the eigenvectors form a basis for \mathbb{R}^2 ? Explain.

Answer:

- a) The characteristic equation is $(1 - \lambda)(-4 - \lambda) - 6 = 0$. This simplifies to $\lambda^2 + 3\lambda - 4 - 6 = 0$ and can be rewritten $(\lambda + 5)(\lambda - 2) = \lambda^2 + 3\lambda - 10 = 0$. The eigenvalues are $\lambda = 2, -5$.

- b) For $\lambda = 2$, the eigenvectors must obey

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

This is satisfied by any non-zero multiple of $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda = 5$, the eigenvectors must obey

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

This is satisfied by any non-zero multiple of $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

- c) Since the eigenvalues are distinct, the eigenvectors must form a basis. Alternatively, the fact that $\det[\mathbf{v}_1 \mathbf{v}_2] = -7$, which is non-zero, shows that we have two linearly independent vectors which must form a basis for \mathbb{R}^2 .

5. Consider the problem of maximizing the function $u(x, y) = x^{1/2} + 3y$ subject to the constraint $x + 2y \leq 10$ and the non-negativity constraints $x \geq 0, y \geq 0$.

- a) Does this problem have a solution? Explain?
- b) If the problem has a solution, use the Kuhn-Tucker theorem to find it. Don't forget to check constraint qualification and the second-order conditions.

Answer:

- a) Yes, it has a solution. We are trying to maximize a continuous function over a compact budget set (we showed in class that such sets are compact). The Weierstrass Theorem tells us there is a solution.

b) There are three constraints. The derivative of the constraints is

$$d\mathbf{g} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Any pair of constraints yields a matrix of rank 2, while a single constraint gives a matrix of rank 1. It follows that constraint qualification (NDCQ) is satisfied.

Note that the Hessian of $x^{-1/2} + 3y$ is always negative semidefinite. The objective is thus concave. It follows that solutions to the first-order conditions will be maxima.

The Lagrangian is $\mathcal{L} = x^{1/2} + 3y - \lambda(x + 2y - 10) + \mu_x x + \mu_y y$. The first-order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2}x^{-1/2} - \lambda + \mu_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 3 - 2\lambda + \mu_y = 0. \end{aligned}$$

Case I: $x, y > 0$. Here $\mu_y = 0$, so $\lambda = 3/2$. As $\mu_x = 0$, $x = 1/9$. The constraint $x + 2y = 10$ binds as $\lambda \neq 0$, so $y = 89/18$.

Case II: $x = 0$ violates the first-order condition as $x^{-1/2}$ is then infinite. This case is impossible.

Case III: $x > 0, y = 0$. Note that $\lambda \geq 3/2$, so the constraint $x + 2y \leq 10$ binds. It follows that $x = 10$ and $y = 0$. But then the x first-order condition yields $\lambda = 1/(2\sqrt{10})$, which contradicts $\lambda \geq 3/2$. This case is also impossible.

It follows that the maximum is at $(1/9, 89/18)$.