Homework Assignment #1

6.3 The economy on the island of Bacchus produces only grapes and wine. The production of 1 pound of grapes requires 1/2 pound of grapes, 1 laborer, and no wine. The production of 1 liter of wine requires 1/2 pound of grapes, 1 laborer, and 1/4 liter of wine. The island has 10 laborers who all together demand 1 pound of grapes and 3 liters of wine for their own consumption. Write out the input-output system for the economy of this island. Can you solve it?

Answer: Let g and w respectively denote the amount of grapes and wine produced. Demand for grapes from wine production is $w/2$, demand from grape production is $g/2$ and demand for consumption is 1. Since demand must equal supply, $g = g/2+w/2+1$. For wine, the corresponding equation is $w = w/4+3$. Finally, for labor, $10 = w + g$. This yields the following input-output system:

$$\begin{aligned} g/2 - w/2 &= 1 \\ 3w/4 &= 3 \\ w + g &= 10 \end{aligned}$$

This is easily solved since the second equation implies $w = 4$. Then the third equation yields $g = 6$, and we have merely to verify that the first equation is also satisfied (otherwise there would be no solution).

6.5 Suppose that 10 percent of white males of working age and 20 percent of black males of working age are unemployed right now. According to Hall's model, what will the corresponding unemployment rates be in the next period?

Answer: Hall's model is given by equations (5) and (6) on page 114. For white males, $x_t = .9$ and $y_t = .1$. Using equations (5), we find that the unemployment rate drops to 8.82%. For black males, $x_t = .8$ and $y_t = .2$. Equations (6) predict the black male unemployment rate drops to 18.28%.

7.11 Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

Answer:

a)

$$\begin{pmatrix} 3 & 3 & | & 4 \\ 1 & -1 & | & 10 \end{pmatrix} \xrightarrow{(2)-3(1)} \begin{pmatrix} 3 & 3 & | & 4 \\ 0 & -2 & | & 26/3 \end{pmatrix}$$

yields row echelon form. Then dividing the second row by -2, we continue until the row echelon form is obtained.

$$\begin{pmatrix} 3 & 3 & | & 4 \\ 0 & 1 & | & -13/3 \end{pmatrix} \xrightarrow{(1)/3} \begin{pmatrix} 1 & 1 & | & 4/3 \\ 0 & 1 & | & -13/3 \end{pmatrix} \xrightarrow{(1)-(2)} \begin{pmatrix} 1 & 0 & | & 17/3 \\ 0 & 1 & | & -13/3 \end{pmatrix}$$

so the solution is $(17/3, -13/3)^T$.

b)

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 6 & 3 & -5 & | & 0 \\ 1 & 1 & 2 & | & 9 \end{pmatrix} \xrightarrow{(2)-3(1)/2} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 0 & -1/2 & | & -3/2 \\ 1 & 1 & 2 & | & 9 \end{pmatrix} \xrightarrow{(3)-(1)/4} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 0 & -1/2 & | & -3/2 \\ 0 & 1/2 & 11/4 & | & 35/4 \end{pmatrix}$$

We now interchange rows (2) and (3) to get row echelon form and continue reducing.

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1/2 & 11/4 & | & 35/4 \\ 0 & 0 & 1/2 & | & -3/2 \end{pmatrix} \xrightarrow{2(2)} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & -1/2 & | & -3/2 \end{pmatrix} \xrightarrow{-2(3)} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\xrightarrow{1(4)} \begin{pmatrix} 1 & 1/2 & -3/4 & | & 1/4 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{(1)-(2)/2} \begin{pmatrix} 1 & 0 & -7/2 & | & -17/2 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\xrightarrow{(1)+7(3)/2} \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{(2)-11(3)/2} \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

which is the reduced row echelon form. The solution is $(2, 1, 3)^T$.

c)

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & -4 & 3 & | & 0 \end{pmatrix} \xrightarrow{(3)-(1)} \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 0 & -6 & 4 & | & -2 \end{pmatrix} \xrightarrow{(2)-(1)/1} \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 0 & 3/2 & | & -3 \\ 0 & -6 & 4 & | & -2 \end{pmatrix}$$

Again, we switch rows (2) and (3) to obtain the row echelon form, then continue.

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix} \xrightarrow{(1)/2} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix} \xrightarrow{-(2)/6} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix}$$

$$\xrightarrow{2(3)/3} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 1 & | & -2 \end{pmatrix} \xrightarrow{(1)+(3)/2} \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}$$

$$\xrightarrow{(2)+2(3)/3} \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix} \xrightarrow{(1)-(2)} \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}$$

which is the reduced row echelon form. The solution is $(1, -1, -2)^T$.

7.15 Use Gauss-Jordan elimination to determine for what values of the parameter k the system

$$x_1 + x_2 = 1$$
$$x_1 - kx_2 = 1$$

has no solutions, one solution, and more than one solution.

Answer: We form the augmented matrix and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -k & 1 \end{pmatrix} \xrightarrow{(2)-1(1)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -k-1 & 0 \end{pmatrix}.$$

Now if $k = -1$, the second line is null. This indicates that the rank of the coefficient matrix is 1, which is less than the number of unknowns (2). Since there is a solution (0,0), there are infinitely many solutions. Here x_2 is a free variable and $x_1 = 1 - x_2$.

However, if $k \neq -1$, we can divide row 2 by $-k - 1$. Then

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{(1)-1(2)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This has the unique solution $x_1 = 1$, $x_2 = 0$.

7.25 For each of the following two systems, we want to separate the variables into exogenous and endogenous ones so that each choice of values for the exogenous variables determines unique values for the endogenous variables. For each system *a)* determine how many variables can be endogenous at any one time, *b)* determine a successful separation into exogenous and endogenous variables, and *c)* find an explicit formula for the endogenous variables in terms of the exogenous ones:

$$i) \quad \begin{aligned} x + 2y + z - w &= 1 \\ 3x + 6y - z - 3w &= 2; \end{aligned} \qquad ii) \quad \begin{aligned} x + 2y + z - w &= 1 \\ 3x - y - 4z + 2w &= 3 \\ y + z + w &= 0. \end{aligned}$$

Answer: There are two equations and four variables in system (i), meaning that we can have at most two exogenous variables unless one of the equations is redundant. We row reduce system (i), obtaining

$$\begin{aligned} x + 2y - w &= 3/4 \\ z &= 1/4. \end{aligned}$$

It is clear that neither equation was redundant. We can group the variables by taking x and z are the endogenous variables and y and w are the endogenous variables. The system above tells us that $z = 1/4$ and $x = 3/4 - 2y + w$.

System (ii) has three equations and four variables, meaning that we can have at most one exogenous variables unless one of the equations is redundant. We row-reduce system (ii), obtaining

$$\begin{aligned} x - z &= 1 \\ y + z &= 0 \\ w &= 0. \end{aligned}$$

None of the equations were redundant, and we can take z as the exogenous variable and x, y, and w as endogenous variables. Then $x = 1 + z$, $y = -z$ and $w = 0$.