Homework Assignment #3

10.10 Find the length of the following vectors. Draw the vectors for a through g.

Answer:

$$a)\ \|(3,4)\| = 5, \quad b)\ \|(0,-3)\| = 3, \quad \|(1,1,1)\| = \sqrt{3},$$

$$d)\ \|3,3)\| = 3\sqrt{2}, \quad \|(-1,-1)\| = \sqrt{2}, \quad \|(1,2,3)\| = \sqrt{14},$$

$$g)\ \|(2,0)\| = 2, \quad \|(1,2,3,4)\| = \sqrt{30}, \quad \|(3,0,0,0,0)\| = 3.$$

10.37 Nonparametric equations of a line in $\mathbb{R}^3$ are equations of the form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \tag{20}$$

These are called **symmetric equations** of the line. They can be derived from the parametric equations by eliminating t, just as one does in the plane.

a) What are the parametric equations which correspond to the symmetric equations (20)?

b) In form (20), one can view the line as the intersection of which two planes?

c) Find the symmetric equations of the following two lines in $\mathbb{R}^3$:

$$i)\ x_1 = 2 - t \qquad\qquad ii)\ x_1 = 1 + 4t$$
$$x_2 = 3 + 4t \qquad\qquad x_2 = 2 + 5t$$
$$x_3 = 1 + 5t; \qquad\qquad x_3 = 3 + 6t.$$

d) For each line in part c, find the equations of two planes whose intersection is that line.

Answer:

a) We call the common ratio in (20) t. Then $x - x_0 = at$, $y - y_0 = bt$ and $z - z_0 = ct$. We can consolidate these equations into $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$.

b) There are several ways to do this. One way is to use the planes $b(x-x_0) = a(y-y_0)$ and $c(y-y_0) = b(z-z_0)$.

c) The symmetric equations are:

$$\frac{2-x_1}{1} = \frac{x_2-3}{4} = \frac{x_3-1}{5}$$

and

$$\frac{x_1-1}{4} = \frac{x_2-2}{5} = \frac{x_3-3}{6}$$

d) One way to do this is to take (i) $8 - 4x_1 = x_2 - 3$ and $10 - 5x_1 = x_3 - 1$ and (ii) $5x_1 - 5 = 4x_2 - 8$ and $6x_2 - 12 = 5x_3 - 15$.

10.41 Use Gaussian elimination to find the equation of the line which is the intersection of the planes $x+y-z = 4$ and $x + 2y + z = 3$.

Answer: The corresponding augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

This yields reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

Thus $x = 5 + 3z$ and $y = -1 - 2z$. We can write this in parametric form as $(x, y, z)^\dagger = (5, -1, 0)^\dagger + t(3, -2, 1)^\dagger$.

11.4 Prove that if (4) holds, then $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$ and $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.

Answer: Line (4) says that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ implies $c_1 = c_2 = 0$. If $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$, we can write $\mathbf{v}_1 = \alpha\mathbf{v}_2$ for some α. Then $\mathbf{v}_1 - \alpha\mathbf{v}_2 = \mathbf{0}$, so by (4), $1 = -\alpha = 0$, which is impossible. This means that $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$.

If $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, there is α with $\mathbf{v}_2 = \alpha\mathbf{v}_1$. But then $-\alpha\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, and (4) tells us that $-\alpha = 1 = 0$, which is impossible. This means that $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.

11.13 Show that the collections in a, b, and d in Example 11.6 form a basis of $\mathbb{R}^2$.

Answer: We can check this by forming the matrix using the vectors as columns, and finding whether the determinant is zero.

$$a)\ \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$$
$$b)\ \det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \neq 0$$
$$d)\ \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0.$$

12.7 Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers that converges to x_0 and that all x_n and x_0 are nonzero.

a) Prove that there is a positive number B such that $|x_n| \geq B$ for all n.

b) Using a, prove that $\{1/x_n\}$ converges to $1/x_0$.

Answer:

a) Since $x_n \to x_0$, we may choose N so that $|x_n - x_0| < |x_0|/2$ for $n \geq N$. Then $|x_0| \leq |x_n| + |x_0 - x_n| \leq |x_n| + |x_0|/2$. Subtracting $|x_0|/2$ from both ends yields $|x_n| > |x_0|/2$ for $n \geq N$. Now let $B = \min\{|x_1|/2, |x_2|/2, \ldots, |x_{N-1}|/2, |x_0|/2\}$. Since each x_n and x_0 are nonzero, $B > 0$. For $n < N$, $|x_n| \geq 2B > B$ by construction while for $n \geq N$, $|x_n| > |x_0|/2 \geq B$ by our original choice of N. This establishes the result.

b) Let $\epsilon > 0$ be arbitrary. Choose N such that $|x_n - x_0| < B|x_0|\epsilon$. Then, for $n \geq N$,

$$\begin{aligned} |x_n - x_0| &= \frac{1}{|x_n|\,|x_0|}|x_n - x_0| \\ &< \frac{1}{|x_n|\,|x_0|}B|x_0|\epsilon \\ &= \frac{B}{|x_n|}\epsilon < \epsilon \end{aligned}$$

where the last inequality follows from part (a). This establishes that $1/x_n \to 1/x_0$.