Homework Assignment #4

13.15 Prove that a linear function from $\mathbb{R}^k$ to $\mathbb{R}^m$ sends a line in $\mathbb{R}^k$ to a point or a line in $\mathbb{R}^m$.

Answer: A line can be written as $\ell = \{\mathbf{x}^0 + t\mathbf{v} : t \in \mathbb{R}\}$. Let L be a linear function from $\mathbb{R}^k \to \mathbb{R}^m$. Then $L(\mathbf{x}^0 + t\mathbf{v}) = L\mathbf{x}^0 + tL\mathbf{v}$. If $L\mathbf{v} = \mathbf{0}$, the line ℓ is mapped to a single point. Otherwise, it is mapped to $\{L\mathbf{x}^0 + tL\mathbf{v} : t \in \mathbb{R}\}$, which is a line in $\mathbb{R}^m$.

13.17 Suppose that $f\colon \mathbb{R}^k \to \mathbb{R}^1$ is a continuous function and that $f(\mathbf{x}^*) > 0$. Show that there is a ball $B = B_\delta(\mathbf{x}^*)$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B$.

Answer: The set $\mathbb{R}_{++} = (0, +\infty)$ is an open set. Since f is continuous, $f^{-1}(\mathbb{R}_{++})$ is open. Moreover, $\mathbf{x}^* \in f^{-1}(\mathbb{R}_{++})$. Now choose $\delta > 0$ so that $B_\delta(\mathbf{x}^*) \subset f^{-1}(\mathbb{R}_{++})$. It has the required property.

14.2 Compute the partial derivatives of the Cobb-Douglas production function $q = kx_1^{a_1}x_2^{a_2}$ and of the Constant Elasticity of Substitution production function $q = k(c_1x_1^{-a} + c_2x_2^{-a})^{-h/a}$, assuming all the parameters are positive.

Answer: For the Cobb-Douglas function,

$$\partial q/\partial x_1 = ka_1x_1^{a_1-1}x_2^{a_2}$$

and

$$\partial q/\partial x_2 = ka_2x_1^{a_1}x_2^{a_2-1}.$$

For the CES production function,

$$\partial q/\partial x_1 = hkc_1x_1^{-1-a}(c_1x_1^{-1} + c_2x_2^{-a})^{-1-h/a}$$

and

$$\partial q/\partial x_2 = hkc_2x_2^{-1-a}(c_1x_1^{-1} + c_2x_2^{-a})^{-1-h/a}.$$

14.8 Use differentials to approximate each of the following:

a) $f(x,y) = x^4 + 2x^2y^2 + xy^4 + 10y$ at $x = 10.36$ and $y = 1.04$;

Answer: $df = (4x^3 + 4xy^2 + y^4)\,dx + (4x^2y + 4xy^3 + 10)\,dy$. We evaluate this at the point $(10,1)$, obtaining $df = 4041\,dx + 450\,dy$. Applying this linear form to $(\Delta x, \Delta y) = (0.36, 0.04)$, we find $\Delta f \approx 4041(.36) + 450(.04) = 1454.76 + 18 = 1472.76$. Since $f(10,1) = 10220$, the approximate value of the function is 11692.76 while the actual value is about 11744.3.

b) $f(x,y) = 6x^{2/3}y^{1/2}$ at $x = 998$ and $y = 101.5$;

Answer: Here we approximate around $(x,y) = (1000,100)$, where $f(1000,100) = 6000$. Then $df = 4x^{-1/3}y^{1/2}\,dx + 3x^{2/3}y^{-1/2}\,dy$. Evaluating at $(1000,100)$, we find $df = 4\,dx + 30\,dy$. Applying this linear form to $(\Delta x, \Delta y) = (-2, 1.5)$, we find $\Delta f \approx -8 + 45 = 37$, so $f(998, 101.5) \approx 6037$. In comparison, the actual value is about 6036.77.

c) $f(x,y,z) = \sqrt{x^{1/2} + y^{1/3} + 5z^2}$ at $x = 4.2$, $y = 7.95$, and $z = 1.02$.

Answer: Here we evaluate around the point $(x,y,z) = (4,8,1)$. Note that $f(4,8,1) = 3$. Then

$$df = \frac{1}{2\sqrt{x^{1/2} + y^{1/3} + 5z^2}}\left[\frac{1}{2}x^{-1/2}\,dx + \frac{1}{3}y^{-2/3}\,dy + 10z\,dz\right].$$

At $(4,8,1)$, this becomes $\frac{1}{6}[\frac{1}{4}\,dx + \frac{1}{12}\,dy + 10\,dz]$. We use $\Delta x = 0.2$, $\Delta y = -0.05$, and $\Delta z = 0.02$. This yields $\Delta f \approx \frac{1}{6}[0.05 - 0.025 + 0.2] = 0.0375$, so $f(4.2, 7.95, 1.02) \approx 3.0375$, compared to the actual value of about 3.041.

14.22 Given that $G(x,y) = (x^2+1, y^2)^{\mathbf{T}}$ and $F(u,v) = (u+v, v^2)^{\mathbf{T}}$, compute the Jacobian derivative matrix of $F\big(G(x,y)\big)$ at the point $(x,y) = (1,1)$.

Answer: By the Chain Rule,

$$d(F \circ G) = dF \times dG = \begin{pmatrix} 1 & 1 \\ 0 & 2v \end{pmatrix} \times \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 0 & 4vy \end{pmatrix}.$$

When $(x,y) = (1,1)$, $(u,v) = G(x,y) = (2,1)$ so the Jacobian becomes

$$d(F \circ G)|_{(x,y)=(1,1)} = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}.$$