Homework Assignment #5

15.18 One solution of the system

$$2x^2 + 3xyz - 4uv = 16, \quad x + y + 3z + u - v = 11$$

is $x = 1$, $y = 2$, $z = 3$, $u = 0$, $v = 1$ (note corrected equation). If one varies u and v near their original values and plugs these new values into this system, can one find unique values of x, y, and z that still satisfy this system? Explain.

Answer: The question is asking whether this system implicitly defines x, y, and z as functions of u and v near $(u, v) = (0, 1)$. In this case, we don't have enough equations to apply the Implicit Function Theorem as there are only 2 equations in 3 unknowns.

Indeed, there are many solutions with $u = 0$ and $v = 1$. These must solve the equations

$$2x^2 + 3xyz = 20, \quad x + y + 3z = 12$$

which reduce to $2x^2 + 12xy - x^2y - xy^2 = 20$, with $z = 4 - (x + y)/3$. Here we can use the Implicit Function Theorem to write y as a function of x because the y-derivative is $12x - x^2 - 2xy$, which is 9 at $(x, y) = (1, 2)$. This means that the system has infinitely many solutions (x, y, z) even at $(u, v) = (0, 1)$.

15.22 Consider the system of equations

$$x + 2y + z = 5, \qquad 3x^2yz = 12,$$

as defining some endogenous variables in terms of some exogenous variables.

a) Divide the three variables into exogenous ones and endogenous ones in a neighborhood of $x = 2$, $y = 1$, $z = 1$ so that the Implicit Function Theorem applies.

Answer: Let F denote the left-hand side of the system. We calculate dF.

$$dF|_{(2,1,1)} = \begin{pmatrix} 1 & 2 & 1 \\ 6xyz & 3x^2z & 3x^2y \end{pmatrix}\Bigg|_{(2,1,1)} = \begin{pmatrix} 1 & 2 & 1 \\ 12 & 12 & 12 \end{pmatrix}$$

We must either treat (x, y) or (y, z) as endogenous to use the Implicit Function Theorem as the matrix will not be invertible if we remove the middle column, but is invertible if we remove either of the other columns.

We choose (x, y) as the endogenous variables and z as the exogenous variable.

b) If each of the exogenous variables in your answer to (*a*) increases by 0.24, use calculus to estimate how each of the endogenous variables will change.

Answer: The Implicit Function Theorem tells us that

$$\begin{pmatrix} \partial x/\partial z \\ \partial y/\partial z \end{pmatrix} = -\begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 12 \end{pmatrix} = -\begin{pmatrix} -1 & \frac{1}{6} \\ 1 & -\frac{1}{12} \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then $\Delta x \approx -1(.24) = -.24$ and $\Delta y \approx 0$.

15.36 Show that the map $F(x, y) = (x + e^y, y + e^{-x})$ is everywhere locally invertible.

Answer: Assuming F is a column vector, we compute

$$dF = \begin{bmatrix} 1 & e^y \\ -e^{-x} & 1 \end{bmatrix}.$$

The determinant is $1 + e^{y-x} > 1 > 0$. Since dF is always invertible, the Inverse Function Theorem guarantees the existence of a local inverse about any point (x_0, y_0).

16.3 Using the method of the previous exercise, sketch a proof that if A is positive (or negative) definite, then every principal submatrix of A is also positive (or negative) definite.

Answer: Suppose A is positive definite and A_k is a principal submatrix of order k. Let $\mathbf{x} \in \mathbb{R}^K$. Define $\mathbf{y}$ by $y_i = 0$ for rows i that are omitted in A_k and succesively fill in the values of $\mathbf{x}$ for the other y_i. Then $\mathbf{x^T} A_k \mathbf{x} = \mathbf{y^T} A \mathbf{y}$. Since A is positive, A_k is also positive. If A is definite, $\mathbf{x^T} A_k \mathbf{x} = 0$ implies $\mathbf{y^T} A \mathbf{y} = 0$, so $\mathbf{y} = \mathbf{0}$. It then follows that $\mathbf{x} = \mathbf{0}$. The negative case is similar.

16.6 Determine the definiteness of the following constrained quadratics:

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

Answer: Here $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1. \end{pmatrix}$$

Here $n = 2$ and $m = 1$, so the only leading principal minor we need check is $H_3 = +2$. As it is positive (same sign as $(-1)^n$), the constrained quadratic is **negative definite**.

b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

Answer: Here $A = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$. The Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Here $n = 2$ and $m = 1$, so the only leading principal minor we need check is $H_3 = -1$. As it is negative (same sign as $(-1)^m$), the constrained quadratic is **positive definite**.

c) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

Answer: Here

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \text{ and } H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{pmatrix}.$$

Here $n = 3$ and $m = 2$, so the only leading principal minor we need check is $H_5 = 16$. This has the same sign as $(-1)^m$, so the constrained quadratic is **positive definite**.

d) $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

Answer: Here

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{pmatrix}.$$

Here $n = 3$ and $m = 2$, so the only leading principal minor we need check is $H_5 = 16$. This has the same sign as $(-1)^m$, so the constrained quadratic is **positive definite**.

e) $Q(x_1,x_2,x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 - x_3 = 0$.

Answer: Here

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & -3 & -1 \end{pmatrix} \text{ and } H = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & -3 \\ -1 & 0 & -3 & -1 \end{pmatrix}.$$

Here $n = 3$ and $m = 1$, so we must check the last 2 leading principal minors $H_3 = 3$ and $H_4 = 4$. Then H_4 has a different sign from $(-1)^n = -1$ and $(-1)^m = -1$. The minors are non-zero and violate the pattern, so the constrained form is **indefinite**.

29.12 For each of the following sets, decide whether or not the set is *a)* a subspace, *b)* closed, *c)* open, *d)* compact, *e)* connected.

$$i)\ \{(x_1,x_2) : x_1 = -x_2\}, \qquad ii)\ \{(x_1,x_2) : 1 < x_1^2 + x_2^2 < 2\},$$
$$iii)\ \{(x_1,x_2,x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}.$$

Answer:

i) This is a straight line through the origin. As a straight line, it **is a vector subspace**. Define $f(x_1,x_2) = x_1 + x_2$, then f is continuous and the set is $f^{-1}(\{0\})$, hence **closed**. It is clear that a ball about the origin cannot be contained in the set, so it is **not open**. It is not bounded, and so is **not compact**. Finally, as a straight line, it **is connected**.

ii) This is an open annulus. It is **not a vector subspace** as $(3/2, 0)$ is in the set, but $(3,0)$ is not. It is **not closed** because $(2 - 1/n, 0)$ is in the set, but the limit $(2,0)$ is not. It **is open** since it is the intersection of two open sets. Let f be the continuous function $f(x_1,x_2) = x_1^2 + x_2^2$ then the set is $f^{-1}(2,+\infty) \cap f^{-1}(-\infty,1)$, which is open. It is not closed, so **not compact**. Finally, the set **is connected** as it is easy to make a path between any two points in it. E.g., from the starting point, continue around the set at the same distance from the origin, until you get to the angle of the end point, then head straight in or out to the end point.

iii) This is a budget set with prices $\mathbf{p} = (1,1,1)$ and income 1. It is **not a vector subspace** because it contains $(1,0,0)$ but not $(2,0,0)$. As a budget set, it is both **closed** and **compact**. It is **not open** as it does not contain any open balls about $(1,0,0)$. Finally, as a convex set, it is **connnected** (in fact, path connected).