Homework Assignment #6

17.3 Which of the maxima and minima in Exercise 17.1 are global maxima or minima?

Answer: Think globally!! All of the functions examined in Exercise 17.1 are polynomials. As such, they can be unbounded above (no global maximum) or unbounded below (no global minimum). Before computing any critical points, you should examine the behavior of the function at infinity to determine whether there can be a global maximum or minimum.

a) Here $f(x, y) = x^4 + x^2 - 6xy + 3y^2$. Since the higher-degree terms are positive, $f(x, y) \to +\infty$ as $\|(x, y)\| \to \infty$. This means there is no global maximum. However, there is a global minimum. In fact, $f(x, y) = x^4 + (x-y)^2 - 4xy + 2y^2$. Since $x^4 \geq 2x^2$ for $|x| \geq \sqrt{2}$, $f(x, y) \geq (x-y)^2 + 2x^2 - 4xy + 2y^2 \geq 3(x-y)^2 \geq 0$ for $|x| \geq \sqrt{2}$. For $|x| \leq \sqrt{2}$, $f(x, y) \geq (x-y)^2 + x^4 + 2(x-y)^2 - 2x^2 \geq 3(x-y)^2 - 4 \geq -4$.

To find any global minima, we examine the critical points.

The derivative is $df = (4x^3 + 2x - 6y, -6x + 6y)$. The Hessian is

$$H = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

This has critical points at $\pm(1, 1)$ and $(0, 0)$. The Hessian is positive definite at $\pm(1, 1)$ (local minima) and is indefinite at $(0, 0)$. Since $f\big(\pm(1, 1)\big) = -1$, both of the local minima are global minima.

b) Here $f(x, y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$. This has neither a global maximum nor a global minimum. We can rewrite this as $f(x, y) = (x-y)^2 - 4xy + 10x + 2y - 5$. Then $\lim_{x\to+\infty} f(x, x) = -\infty$ and $\lim_{x\to+\infty} f(x, -x) = +\infty$.

c) Here $f(x, y) = xy^2 + x^3y - xy$. Then $\lim_{x\to+\infty} f(x, x) = +\infty$ and $\lim_{x\to+\infty} f(x, -x) = -\infty$, so there are no global maxima or minima.

d) Here $f(x, y) = 3x^4 + 3x^2y - y^3$. Then $\lim_{y\to+\infty} f(0, y) = +\infty$ and $\lim_{y\to-\infty} f(0, y) = -\infty$, so there are no global maxima or minima.

17.7 For the discriminating monopolist of Example 17.3, compute the demand function for the market as a whole, without price discrimination. Compute the firm's profit-maximizing output for this situation and compare the profit to the computation in Example 17.3.

Answer: Without price discrimination, the price is the same (p) in both markets. Quantity demanded obeys $p = 50 - 5Q_1$ in market 1 and $p = 100 - 10Q_2$ in market 2. Thus

$Q_1 = 10 - p/5$ and $Q_2 = 10 - p/10$. Total quantity demanded is $Q = Q_1 + Q_2 = 20 - 3p/10$. Total profit is then

$$\begin{aligned} p(Q_1 + Q_2) - \big(90 + 20(Q_1 + Q_2)\big) &= 20p - 3p^2/10 - 90 - 400 + 6p \\ &= -490 + 26p - 3p^2/10. \end{aligned}$$

Since the second derivative of profit is negative, we need only check the first-order condition to find the profit-maximizing price p^*. Thus $26 - 6p^*/10 = 0$, so $p^* = 130/3$. The corresponding profit-maximizing quantities are $Q_1^* = 4/3$ and $Q_2^* = 17/3$, so $Q^* = 7$. It follows that profit is $220/3$.

Since total production is the same as in the example, the difference in profit is due to the difference in total revenue. Under the single-price regime, revenue is $7(130/3) = 910/3 = 303\frac{1}{3}$. Under the two-price regime, revenue in market one is $35(3) = 105$, and revenue in market two is $60(4) = 240$, for a total of 345. Thus profit is $42\frac{1}{3}$ higher under the two-price regime.

18.6 Find the max and min of $f(x, y, z) = x + y + z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $y = 0$.

Answer: We will do it the long way. The derivative of the constraints is

$$\begin{bmatrix} 2x & 2y & 2z \\ 0 & 1 & 0 \end{bmatrix}.$$

Constraint qualification is satisfied because at least one of x and z is non-zero, yielding rank 2.

The Lagrangian is $L = x + y + z^2 - \mu(x^2 + y^2 + z^2 - 1) - \nu y$. The first-order conditions are $0 = 1 - 2\mu x$, $0 = 1 - 2\mu y - \nu$, and $0 = 2z - 2\mu z$. Since $y = 0$, the second equation becomes $\nu = 1$. The third equation is $2(1 - \mu)z = 0$.

If $z = 0$, we use the constraints to find either $x = 1$ and $\mu = 1/2$ or $x = -1$ and $\mu = -1$.

If $z \neq 0$, $\mu = 1$, and $x = 1/2$. Then the constraints imply either $z = \sqrt{3}/2$ or $z = -\sqrt{3}/2$.

Now $f(1, 0, 0) = 1$, $f(-1, 0, 0) = -1$, and $f(1/2, 0, \sqrt{3}/2) = f(1/2, 0, -\sqrt{3}/2) = 5/4$. Thus $(-1, 0, 0)$ is the minimum and $(1/2, 0, \sqrt{3}/2)$ and $(1/2, 0, -\sqrt{3}/2)$ are both maxima.

This can be simplified by using $y = 0$ to reduce it to a problem involving only x and z.

18.15 Maximize $3xy - x^3$ subject to the constraints $2x - y = -5$, $5x + 2y \geq 37$, $x \geq 0$, $y \geq 0$.

Answer: We start by considering the constraints. One of them is redundant. The equality constraint tells us that $y = 2x+5$. Substituting in the first inequality constraint, we find $9x + 10 \geq 37$, so $x \geq 3 > 0$. The third constraint is redundant. Further, since $y = 2x + 5 \geq 11$, the fourth constraint is also redundant.

There are now two ways to solve the problem. The first is to substitute $2x + 5$ for y. In that case, we must maximize $g(x) = 6x^2 + 15x - x^3$ subject to the constraint $x \geq 3$. Now $g'(3) = 24 > 0$, so 3 is not a maximum. g eventually becomes negative as x increases, so we check the critical points, setting $g' = 12x + 15 - 3x^2 = 0$. Now $3x^2 - 12x - 5 = 3(x + 1)(x - 5)$, so $x = 5$ is the only critical point with $x \geq 3$, which must be the maximum. Thus $(x, y) = (5, 15)$ is the maximum for the original problem.

Method two is to use the Lagrangian

$$L = 3xy - x^3 - \mu(2x - y + 5) - \lambda_1(5x + 2y - 37) - \lambda_2 y.$$

The first-order conditions are

$$0 = 3y - 3x^2 - 2\mu - 5\lambda_1$$
$$0 = 3x - \mu - 2\lambda_1.$$

The derivative of the possibly binding constraints is

$$\begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$$

which has rank 2. Constraint qualification will be satisfied whether or not the second constraint binds.

We first consider the case where the second constraint binds: $5x + 2y = 37$. Using the constraint $2x - y = -5$, we find that $(x, y) = (3, 11)$. The first-order conditions become

$$0 = 6 + 5\lambda_1 - 2\mu$$
$$0 = 9 + 2\lambda_1 + \mu.$$

This yields $\lambda_1 = -8/3$, violating non-negativity.

Then $\lambda_1 = 0$ by complementary slackness and the first-order conditions are:

$$0 = 3y - 3x^2 - 2\mu$$
$$0 = 3x - \mu.$$

Using $y = 2x + 5$, we find $3x^2 - 12x - 15 = 0$ as before, yielding a maximum at $(x, y) = (5, 15)$ with $\lambda_2 = 0$.

19.14 Check the second-order conditions for the solutions of the first-order conditions in Exercises 18.2, 18.3, and 18.5.

Answer:

18.2 The bordered Hessian of the Lagrangian is:

$$\begin{bmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2+2\mu & \mu \\ 2y+x & \mu & 2+2\mu \end{bmatrix}.$$

There is one constraint ($k = 1$) and two variables ($n = 2$), so we look only at the determinant of the full bordered Hessian.

When $\mu = -2$ and $y = -x$, the determinant is $8x^2 > 0$. Since $(-1)^n H_3 > 0$, the solutions $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ are local maxima.

When $\mu = -2/3$ and $y = x$, the determinant is $-24x^2 < 0$. Since $(-1)^k H_3 > 0$, the solutions $(1, 1)$ and $(-1, -1)$ are local minima.

18.3 The bordered Hessian of the Lagrangian is:

$$\begin{bmatrix} 0 & 2x & 1 \\ 2x & 2-2\mu & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

There is one constraint ($k = 1$) and two variables ($n = 2$), so we look only at the determinant of the full bordered Hessian.

The determinant is $2\mu - 2 - 8x^2$. Plugging in $x = 1.17$ and $\mu = -0.7$, $H_3 < 0$. This is the same sign as $(-1)^k$, so we have a local minimum.

18.5 The unbordered Hessian of the Lagrangian is:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This is a positive definite matrix, and so we have a local minimum.

19.21 Which of the last three constraint qualifications in Theorem 19.12 hold for the constraint functions in Exercises 18.10, 18.11, 18.12, 18.17 and 18.18?

Answer: The three conditions in question are (c) Slater's condition: There is a ball $U \subset \mathbb{R}^n$ with $\mathbf{x}^* \in U$ where $g_1, \ldots, g_h$ are convex functions on U and there exists $\mathbf{z} \in U$ with $g_i(\mathbf{z}) < b_i$ for each i; (d) concavity: $g_1, \ldots, g_h$ are concave functions; (e) linearity: $g_1, \ldots, g_h$ are linear functions.

18.10 (c), (d), (e). The constraints are linear, which implies (e) and (d). Slater's condition (c) is also easily verified.

18.11 (c) only. One of the constraints is not linear, so (e) fails. The first constraint is not concave, so (d) fails. However, Slater's condition (c) is easily checked at all boundary points.

18.12 (c) only. The first constraint, $x^2 + y^2 + z \leq 6$, is neither linear nor is the left-hand side a concave function, so (d) and (e) fail. Slater's condition is easily satisfied in the interior, so we need only worry about the boundary points. At the boundary points, there are always nearby points where no constraint binds and the functions g_i are convex, so (c) holds.

18.17 (c) only. Note that 18.17 is a minimization problem. This can be converted to a maximization problem by maximizing $-f = 2y - x^2$. The constraints are unaffected, so we apply the same criteria as before. Two of the constraints are linear, while $g_1(\mathbf{x}) = x^2 + y^2 \leq 1$ is not, so (e) fails. Further g_1 is convex, but **not concave**, so (d) fails. Finally, Slater's condition is satisfied (c). To see this note that the constraint set is the upper right-hand quadrant of the unit disk. Any point in the set has nearby points where no constraint binds.

18.18 (c), (d), (e). This is also a minimization problem, only the constraint $4x + 3y \leq 10$ binds at the optimum. Since it is linear, both (d) and (e) are satisfied. It is also easily verified that Slater's condition (c) holds.