# Mathematical Economics Exam #2, Nov. 9, 2017

1. Consider the problem of maximizing $p_1x_1 + p_2x_2$ under the constraint $x_1^4 + x_2^4 \le 1$. The parameters obey $p_1p_2 \ne 0$.

   *a*) Is constraint qualification satisfied?

   **Answer:** There is only one constraint, with derivative $dg = (4x_1^3, 4x_2^3)$. This is non-zero unless $\mathbf{x} = \mathbf{0}$. As there is is one constraint, the non-degenerate constraint qualification condition (NDCQ) is satisfied for $\mathbf{x} \ne \mathbf{0}$. When the constraint binds, $\mathbf{x} \ne \mathbf{0}$, so NDCQ is satisfied.

   *b*) Find all solutions to the maximization problem.

   **Answer:** Form the Lagrangian $\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(1 - x_1^4 - x_2^4)$. The first order conditions are

$$0 = p_1 - 4\lambda x_1^3$$
$$0 = p_2 - 4\lambda x_2^3$$

   It follows that $\lambda > 0$, so $x_1^4 + x_2^4 = 1$ (the constraint binds by complementary slackness). Then $p_1/p_2 = x_1^3/x_2^3$, so $x_2^4 = (p_2/p_1)^{4/3}x_1^4$. Substituting into the constraint, we find $[1+(p_2/p_1)^{4/3}]x_1^4 = 1$. Since $\lambda > 0$, $x_i$ must have the same sign as $p_i$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{(p_1^{4/3} + p_2^{4/3})^{1/4}} \begin{pmatrix} (\operatorname{sgn} p_1)p_1^{1/3} \\ (\operatorname{sgn} p_2)p_2^{1/3} \end{pmatrix}.$$

   It follows that the maximum value of $p_1x_1 + p_2x_2$ is

$$\left(p_1^{4/3} + p_2^{4/3}\right)^{3/4}.$$

   Although it is beyond the scope of this problem, we can now obtain a more general result. For $1 \le p < \infty$, define $\|\mathbf{x}\|_p$ by

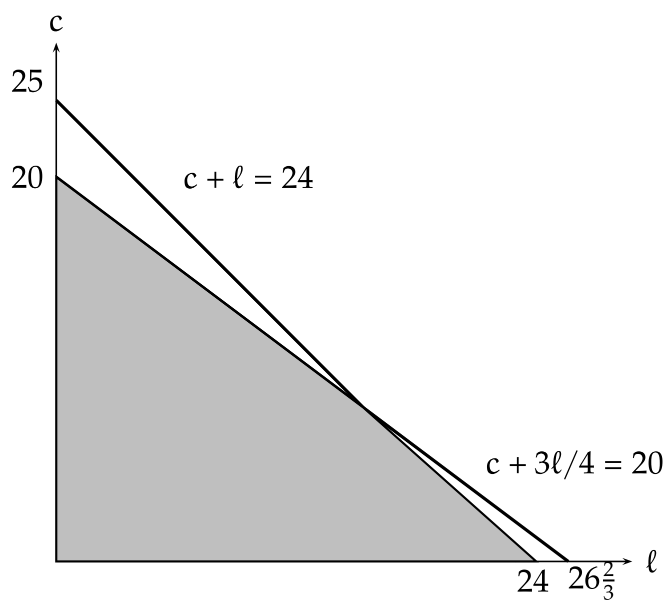
$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{L} |x_i|^p\right)^{1/p}.$$

   The same technique shows that in $\mathbb{R}^L$, maximizing $\mathbf{p}\cdot\mathbf{x}$ subject to the constraint $\|\mathbf{x}\|_p \le 1$ for $1 < p < \infty$, yields maximium value $\|\mathbf{p}\|_q$ with $1/p + 1/q = 1$. Thus for any $x \neq \mathbf{0}$ $\mathbf{p}\cdot(x/\|x\|_p) \leq \|\mathbf{p}\|_q$, implying that $\mathbf{p}\cdot x \leq \|\mathbf{p}\|_q\|x\|_p$. Consideration of $-x$ allows us to write this as $|\mathbf{p}\cdot x| \leq \|\mathbf{p}\|_q\|x\|_p$, which is a special case of Hölder's Inequality.

2. Consider the problem of maximizing $\sqrt{c\ell}$ under the constraints $c+\ell \leq 24$, $c+3\ell/4 \leq 20$, $c \geq 0$, and $\ell \geq 0$.

   *a*) Sketch the feasible set.

   **Answer:**




   *b*) Is constraint qualification satisfied?

   **Answer:** Two of the constraints must be rewritten as $-c \leq 0$ and $-\ell \leq 0$. Then

$$d\mathbf{g} = \begin{pmatrix} 1 & 1 \\ 1 & 3/4 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Each line is non-zero, so if only one constraint binds, constraint qualification is satisfied. Part (a) makes it clear that at most two constraints can bind. Any pair of rows of $d\mathbf{g}$ are linearly independent, so the rank is two in any of these cases. This means that the non-degenerate constraint qualification condition (NDCQ) is satisfied.

   *c*) Find all solutions to the maximization problem.

**Answer:** The Lagrangian is $\mathcal{L} = \sqrt{c\ell} + \lambda_1 c + \lambda_2 \ell + \lambda_3(24 - c - \ell) + \lambda_4(20 - c - 3\ell/4)$. The first order conditions are

$$0 = \sqrt{\ell/c} + \lambda_1 - \lambda_3 - \lambda_4$$
$$0 = \sqrt{c/\ell} + \lambda_2 - \lambda_3 - 3\lambda_4/4.$$

These conditions cannot be satisfied if either $c = 0$ or $\ell = 0$. By complementary slackness, $\lambda_1 = \lambda_2 = 0$. Complementary slackness also shows that at least one of constraints 3 and 4 must bind. We now rewrite the first order conditions

$$\sqrt{\ell/c} = \lambda_3 + \lambda_4$$
$$\sqrt{c/\ell} = \lambda_3 + 3\lambda_4/4.$$

First, consider the case $c + \ell < 24$. Then $\lambda_3 = 0$. The first order conditions can be reduced to $\ell/c = 4/3$. Further, since $\lambda_4 > 0$, $c + 3\ell/4 = 20$, so $c = 10$ and $\ell = 40/3$. Then $c + \ell = 232/3 < 24$, so all the constraints are satisfied.

Next, suppose $c + 3\ell/4 < 20$. Then $\lambda_4 = 0$, and we have $\ell/c = 1$. Since $\lambda_3 > 0$, $c + \ell = 24$, implying $c = \ell = 12$. But then $c + 3\ell/4 = 21 > 21$, contradicting our assumption that $c + 3\ell/4 < 20$. There is no solution here.

The only solution is $(c, \ell) = (10, 40/3)$.

3. Consider the set $A = \{(x, y, z) : x + 2y + 3z = 6\}$.

   *a*) Is $A$ a vector subspace? Explain.

   *b*) Is $A$ open? Explain.

   *c*) Is $A$ connected? Explain.

   *d*) Is $A$ closed? Explain.

   *e*) Is $A$ compact? Explain.

**Answer:** The set $A$ is a plane that does not go through the origin.

*a*) **No.** The point $(1, 1, 1) \in A$, but $2(1, 1, 1) \notin A$, so it is not a vector subspace.

*b*) **No.** The point $(1, 1, 1) \in A$, but $(1 + \varepsilon, 1, 1) \notin A$ for $\varepsilon > 0$. This means any $B_\varepsilon(1, 1, 1)$ is not contained in $A$.

*c*) **Yes.** Suppose $(x, y, z), (x', y', z') \in A$. Consider the path defined by $f(t) = t(x', y', z') + (1 - t)(x, y, z)$. Then $f(0) = (x, y, z)$, $f(1) = (x', y', z')$ and $f(t) \in A$ for all $t \in [0, 1]$. This means that $A$ is path-connected, hence connected.

*d*) **Yes.** The set is closed because $f(x, y, z) = x + 2y + 3z$ is continuous and $A$ is the inverse image of a closed set, $A = f^{-1}\{6\}$.

*e*) **No.** The set is not bounded because $(6, n, -n) \in A$ for all $n$.

4. Let $f(x, y, z) = x^2 + y^2 + 3z$.

*a*) Does this function map $\mathbb{R}^3$ onto $\mathbb{R}$?

**Answer: Yes**, it is onto. In fact, we can even set two of the variables to zero. Here $f(0, 0, a/3) = a$, showing that the function takes all real values.

*b*) Find all points $(x_0, y_0, z_0)$ satisfying $f(x_0, y_0, z_0) = 12$.

**Answer:** Such points obey $x_0^2 + y_0^2 = 12 - 3z_0$. This only has a solution when $12 - 3z_0 \geq 0$. Thus the solutions are all $(x_0, y_0, z_0)$ obeying $x_0^2 + y_0^2 = 12 - 3z_0$ with $z_0 \leq 4$.

*c*) For what $(x_0, y_0, z_0)$ obeying $f(x_0, y_0, z_0) = 12$ is there a differentiable function $g(y, z)$ defined on some neighborhood of $(y_0, z_0)$ that obeys $x_0 = g(y_0, z_0)$ and $f\big(g(y, z), y, z\big) = 12$?

**Answer:** Since $\partial f/\partial x = 2x$, $(\frac{\partial f}{\partial x})(x_0, y_0, z_0) = 2x_0$. The Implicit Function Theorem yields such a function $g$ whenever $x_0 \neq 0$, which requires $z_0 < 4$. Alternatively, note that $g(y, z) = (\operatorname{sgn} x_0)(12 - 3z_0 - z^2)^{1/2}$ works.

*d*) Compute $dg$ at $(x_0, y_0, z_0)$ obeying $f(x_0, y_0, z_0) = 12$.

**Answer:** By the Implicit Function Theorem,

$$\begin{aligned} dg &= -\frac{1}{2x}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= -\left(\frac{y_0}{x_0}, \frac{3}{2x_0}\right). \end{aligned}$$