Homework Assignment #1

6.1 Suppose that the firm in Example 1 did not make any charitable contribution. Write out and solve the system of equations which describe its state and federal taxes. What is the net cost of its $5956 charitable contribution?

Answer: Profits are $100,000. State taxes are 5 percent of profits, or $5,000. Federal taxes are levied on profits minus state taxes and charitable contributions. Since there are no charitable contributions, federal taxes are 40 percent of $95,000, or $38,000. This means the firm earns $57,000 in after-tax profits.

In comparison, profits after taxes and contributions in Example 1 are $53,605. Thus after tax and contribution profits have fallen by $3,395 as a result of the $5,956 charitable contribution.

6.6 For the Markov employment model, Hall gives $p = .106$ and $q = .993$ for black females, and $p = .151$ and $q = .997$ for white females. Write out the Markov systems of difference equations for these two situations. Compute the stationary distributions.

Answer: For black females, the Markov system is:

$$\begin{aligned} x_{t+1} &= .993x_t + .106y_t \\ y_{t+1} &= .007x_t + .894y_t \end{aligned}$$

and for white females, the Markov system is:

$$\begin{aligned} x_{t+1} &= .997x_t + .151y_t \\ y_{t+1} &= .003x_t + .849y_t \end{aligned}$$

where x_t and y_t denote the average number employed and unemployed, respectively.

The stationary distribution is given by

$$x = \frac{p}{1 + p - 1} \qquad \text{and} \qquad y = \frac{1 - q}{1 + p - q}.$$

In the case of black females, this yields $x = .938$ and $y = .062$, while for white females, $x = .981$ and $y = .019$.

7.7 Use Gaussian elimination to solve

$$\begin{aligned} 3x + 3y &= 4 \\ -x - y &= 10 \end{aligned}$$

What happens and why?

Answer: We start by normalizing the top equation (divide it by 3).

$$\begin{aligned} x + y &= 4/3 \\ -x - y &= 10 \end{aligned}$$

Then we add the top equation to the bottom equation to eliminate the x term:

We can see that there is no solution. The two original equations are inconsistent with each other. Had we converted it to matrix form, we would find the rank of the coefficient matrix is 1 while the rank of the augmented matrix is 2. Since the augmented matrix has higher rank, there is no solution.

7.12 Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x - z &= 0 \end{aligned}$$

Answer: We form the augmented matrix and row-reduce.

$$\begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 2 & 3 & 7 & -2 & 9 \\ 3 & 5 & 13 & -9 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{(2)-2(1)} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 3 & 5 & 13 & -9 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{(3)-3(1)} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{(4)+2(1)} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ 0 & 3 & 6 & -5 & 0 \end{pmatrix}$$

$$\xrightarrow{(3)-2(2)} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 3 & 6 & -5 & 0 \end{pmatrix} \xrightarrow{(4)-3(2)} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix}$$

$$\xrightarrow{(1)-1(2)} \begin{pmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix} \xrightarrow{(3)/2} \begin{pmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix}$$

$$\xrightarrow{(1)-2(3)} \begin{pmatrix} 1 & 0 & 0 & 3 & 8 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix} \xrightarrow{(2)-(3)} \begin{pmatrix} 1 & 0 & 0 & 3 & 8 \\ 0 & 1 & 0 & 11/2 & 35/2 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix}$$

$$\xrightarrow{(4)-3(3)} \begin{pmatrix} 1 & 0 & 0 & 3 & 8 \\ 0 & 1 & 0 & 11/ & 35/2 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 0 & -1/2 & -3/2 \end{pmatrix} \xrightarrow{2(4)} \begin{pmatrix} 1 & 0 & 0 & 3 & 8 \\ 0 & 1 & 0 & 11/2 & 35/2 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{(1)-3(4)} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 11/2 & 35/2 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{(2)-11(4)/2} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{(2)+7(4)/2} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

The solution is $(w, x, y, z) = (-1, 1, 2, 3)$.

7.29 Consider the system

$$\begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + 2z &= 3 \\ 3w + 7x + y + z &= 6 \end{aligned}$$

a) Separate the variables into endogenous and exogenous ones so that each choice of the exogenous variables uniquely determines values for the endogenous ones.

b) For you answer to *a*, what are the values of the endogenous variables when all the exogenous variables are set to zero?

c) Find a separation into endogenous and exogenous variables (same number of each as in part *a*) that will not work in the sense of *a*. Find a value of the new exogenous variables for which there are infinitely many corresponding values of the endogenous variables.

Answer:

a) Taking w as exogenous and x, y, z as endogenous will do the trick. In this case the matrix of the corresponding columns is

$$B = \begin{pmatrix} -1 & 3 & -1 \\ 4 & -1 & 2 \\ 7 & 1 & 1 \end{pmatrix}.$$

Checking, we find B has rank 3, so the Linear Implicit Function Theorem gives the result.

b) Setting $w = 0$ yields augmented matrix

$$\hat{B} = \begin{pmatrix} -1 & 3 & -1 & 0 \\ 4 & -1 & 2 & 3 \\ 7 & 1 & 1 & 6 \end{pmatrix}.$$

We can row-reduce this to

$$\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 11 & -2 & 3 \\ 0 & 0 & -10 & 0 \end{pmatrix}.$$

At this point the easiest way to solve it is to note that $z = 0$, implying $y = 3/11$ and $x = 9/11$.

c) Taking z as exogenous will work. The matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & -1 \\ 1 & 4 & -1 & 2 \\ 3 & 7 & 1 & 1 \end{pmatrix}$$

can be row-reduced to

$$\begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 5 & -4 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Consider the homogeneous system and set $z = 0$. The solutions to this obey $5x = 4y$ and $w - x = -3y$. Then for every value y there will be w and x that solve the system, yielding infinitely many solutions. [In this case the rank of the matrix of endogenous variables is 2, which is less than the number of equations.]