Homework Assignment #2

8.5 It sometimes happens that $AB = BA$.

a) Check this for $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and $B = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$.

b) Show that if B is a scalar multiple of the 2×2 identity matrix, then $AB = BA$ for all 2×2 matrices A.

Answer:

a) Here $AB = BA = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}$.

b) Here $B = \lambda I_2$, so $AB = A(\lambda I_2) = \lambda(AI_2) = \lambda A = 2(I_2 A) = (\lambda I_2 A) = BA$.

8.25

a) Prove Theorem 8.10, which states that when A and B are square invertible matrices

- *a*) $(A^{-1})^{-1} = A$.
- *b*) $(A^{\mathbf{T}})^{-1} = (A^{-1})^{\mathbf{T}}$.
- *c*) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

b) Generalize part (*c*) to the case of the product of k nonsingular matrices.

c) Show by example that if A and B are invertible, $A + B$ need not be invertible.

d) Show that, when it exists, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$.

Answer:

a)

- *a*) Since $AA^{-1} = I = A^{-1}A$, $A = (A^{-1})^{-1}$.
- *b*) Taking the transpose of $AA^{-1} = I = A^{-1}A$, we obtain $(A^{-1})^T A^T = I = A^T (A^{-1})^T$ which shows $(A^T)^{-1} = (A^{-1})^T$.
- *c*) $(B^{-1}A^{-1})(AB) = B^{-1}B = I = (AB)(B^{-1}A^{-1})$, showing that $(AB)^{-1} = B^{-1}A^{-1}$.

b) If $A_1, \ldots, A_k$ are non-singular $n \times n$ matrices, then $(A_1 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}$.

c) Set $A = I$ and $B = -I$. Both are invertible, but $A + B = 0$ which is not invertible

d) This is false even for 1×1 matrices! E.g., $1/2 + 1/2 \neq 1/4$.

9.8 Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \qquad b) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

Answer: We row reduce the first matrix by first subtracting the first row from the second and third rows, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, and then subtract the second row from the first, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. As with all row-echelon matrices, we now find the determinant by multiplying the diagonal terms. Thus the matrix in (a) has determinant 3.

For the second matrix, first subtract row 1 from row 3, and then subtract twice row 2 from row 3, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{pmatrix}$. This has determinant -20.

9.17 If we introduce tax rate t and let the consumption function depend on after-tax income $C = b(Y - tY)$, then system (10) becomes

$$\begin{aligned} (1-t)sY + ar &= I^o + G \\ mY - hr &= M_s - M^o \end{aligned}$$

Use Cramer's Rule to see how the equilibrium Y and r are affected by the tax rate t.

Answer: By Cramer's Rule

$$\begin{aligned} Y &= \begin{vmatrix} I^o + G & a \\ M_s - M^o & -h \end{vmatrix} \Big/ \begin{vmatrix} (1-t)s & a \\ m & -h \end{vmatrix} = \frac{h(I^o + G) + a(M_s - M^o)}{(1-t)sh + am} \\ r &= \begin{vmatrix} (1-t)s & I^o + G \\ m & M_s - M^o \end{vmatrix} \Big/ \begin{vmatrix} (1-t)s & a \\ m & -h \end{vmatrix} = \frac{m(I^o + G) - (1-t)s(M_s - M^o)}{(1-t)sh + am} \end{aligned}$$

To see the influence of t, take the partial derivatives. We find

$$\begin{aligned} \frac{\partial Y}{\partial t} &= (sh)\frac{h(I^o + G) + a(M_s - M^o)}{\big((1-t)sh + am\big)^2} = \frac{shY}{(1-t)sh + am} > 0 \\ \frac{\partial r}{\partial t} &= s(M_s - M^o)\left[\frac{1}{(1-t)sh + am} - \frac{m(I^o + G) - (1-t)s(M_s - M^o)}{\big((1-t)sh + am\big)^2}\right] \\ &= \frac{s(M_s - M^o)}{(1-t)sh + am}(1 - r) > 0. \end{aligned}$$

Both Y and r increase when the tax rate t increases.

26.22 Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

Answer:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(a+b)(c-a) \end{vmatrix}$$

The determinant is then $(b-a)[c^2-a^2-(a+b)(c-a)] = (b-a)(c-a)[a+c-a-b] = (b-a)(c-a)(c-b)$.