Homework Assignment #3

10.27 Show that the midpoint of $\ell(\mathbf{x}, \mathbf{y})$ occurs where $t = \frac{1}{2}$. In other words, if $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$, show that $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{y} - \mathbf{z}\|$.

Answer: Here $\|\mathbf{x} - \mathbf{z}\| = \|\frac{1}{2}\mathbf{x} - \frac{1}{2}\mathbf{y}\|$ and $\|\mathbf{y} - \mathbf{z}\| = \|\frac{1}{2}\mathbf{y} - \frac{1}{2}\mathbf{x}\|$. Since the norm is absolutely homogeneous of degree 1, the two expressions are equal.

10.31 Transform each of the following nonparametrized equations into form (10):

$$a)\ 2x_2 = 3x_1 + 5; \qquad b)\ x_2 = -x_1 + 7; \qquad c)\ x_1 = 6.$$

Answer:

a) $\mathbf{x} = (0, 5/2) + t(1/3, 1/2)$.

b) $\mathbf{x} = (7, 0) + t(-1, 1)$.

c) $\mathbf{x} = (6, 0) + t(0, 1)$.

11.3 Determine whether or not each of the following collections of vectors in $\mathbb{R}^4$ are linearly independent.

$$a)\ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}; \quad b)\ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Answer:

a) Let the vectors be $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. We will have $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$ if and only if $x_1 + x_2 = 0$, $x_1 + x_3 = 0$, $x_2 + x_3 = 0$. The first two equations imply $x_2 = x_3$, so the third tells us $x_2 = x_3 = 0$. Then subsitute in the first to find $x_1 = 0$. Since the x's must be zero, the vectors are linearly independent.

b) Let the vectors be $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 = \mathbf{0}$, which means the vectors are linearly dependent.

11.14 Which of the following are bases of $\mathbb{R}^3$?

$$a)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}; \quad b)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}; \quad c)\ \begin{pmatrix}6\\3\\9\end{pmatrix}, \begin{pmatrix}5\\2\\8\end{pmatrix}, \begin{pmatrix}4\\1\\7\end{pmatrix};$$

$$d)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad e)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Answer:

a) No, not a basis. The set does not span. There need to be 3 basis vectors in $\mathbb{R}^3$.

b) No, not a basis. The vectors are not linearly independent as $2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$.

c) No, not a basis. The matrix A formed by the column vectors has $\det A = 0$, so it is not invertible. The vectors neither span, nor are linearly independent.

d) Yes, it is a basis. There are 3 vectors and the matrix A formed by the column vectors has $\det A = -1$, so it is invertible.

e) No, not a basis. The set is not linearly independent since $\mathbf{v}_1 + \mathbf{v}_4 = \mathbf{v}_2$. There need to be 3 basis vectors in $\mathbb{R}^3$.

12.6 Prove that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences with limits x and y, respectively, then the sequence $\{x_n - y_n\}_{n=1}^{\infty}$ converges to the limit $x - y$.

Answer: Consider

$$\begin{aligned} |(x_n - y_n) - (x - y)| &= |(x_n - x) - (y_n - y)| \\ &\leq |x_n - x| + |y_n - y|. \end{aligned}$$

Let $\epsilon > 0$. Choose N_1 such that $|x_n - x| < \epsilon/2$ for $n > N_1$. Then choose $N_2 \geq N_1$ with $|y_n - y| < \epsilon/2$ for $n > N_2$. It follows that for $n > N_2 \geq N_1$,

$$|(x_n - y_n) - (x - y)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

establishing convergence.