Homework Assignment #5

15.1

a) Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$.

b) Then, estimate the y value which corresponds to $x = 4.8$.

Answer:

a) Let $f(x, y) = x^2 - xy^3 + y^5 - 17$. Then $\partial f/\partial y = -3xy^2 + 5y^4$. Evaluating at $(x, y) = (5, 2)$ we find $\partial f/\partial y(5, 2) = -60 + 80 = 20$. Since this is non-zero, the implicit function theorem tells us that we may write y as a function of x in a neighborhood of $x = 5$.

b) Now $dy/dx = -\partial f/\partial x \Big/ \partial f/\partial y$. Evaluating at $(x, y) = (5, 2)$, we obtain $dy/dx = -2/20 = -1/10$. Since $\Delta x = -0.2$, $\Delta y \approx +0.02$, yielding a value of approximately $y = 2.02$.

15.12 Consider the function $f(x, y) = x^2 e^y$.

a) What is the slope of the level set at $x = 2$, $y = 0$?

Answer: The equation $f(x, y) = c$ implicitly defines y as a function of x. Here $c = f(2, 0) = 4$. Since $df(x, y) = (2xe^y, x^2e^y)$, $df(2, 0) = (4, 4)$. Since the second component is non-zero, the implicit function theorem tells us that we can write $y = y(x)$ in a neighborhood of $(2, 0)$ and that $dy/dx = -(2xe^y)/(x^2e^y) = -2/x$. Evaluating at $(2, 0)$, we find $y'(2) = -1$, which is the slope of the level set.

b) In what direction should one move from the point $(2, 0)$ in order to increase f most quickly? Express your answer as a vector of length 1.

Answer: Here $\nabla f = (2xe^y, x^2e^y)^{\mathbf{T}}$. The direction is given by $\nabla f(2, 0) = (4, 4)^{\mathbf{T}}$. We normalize, obtaining $(1/\sqrt{2})(1, 1)^{\mathbf{T}}$.

15.20 Check that $x = 1$, $y = 4$, $u = 1$, $v = -1$ is a solution of the system

$$y^2 + 2u^2 + v^2 - xy = 15, \qquad 2y^2 + u^2 + v^2 + xy = 38.$$

If y increases to 4.02 and x stays fixed, does there exist a (u, v) near $(1, -1)$ which solves this system? Why or why not? If yes, estimate the new u and v.

Answer: It is easily verified that $(1, 4, 1, -1)$ solves the system.

Let

$$F(x, y, u, v) = \begin{pmatrix} y^2 + 2u^2 + v^2 - xy - 15 \\ 2y^2 + u^2 + v^2 + xy - 38 \end{pmatrix}.$$

Then

$$dF_{(u,v)} = \begin{bmatrix} 4u & 2v \\ 2u & 2v \end{bmatrix},$$

so

$$dF_{(u,v)}(1, 4, 1, -1) = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix}.$$

As this is invertible, the implicit function theorem says that there is a solution to the system near $(1, 4, 1, -1)$. The y derivatives of u and v are given by

$$\begin{pmatrix} \partial u/\partial y \\ \partial v/\partial y \end{pmatrix} = -\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1 \end{bmatrix} \begin{pmatrix} 7 \\ 17 \end{pmatrix} = \begin{pmatrix} 5 \\ 27/2 \end{pmatrix}.$$

Since $\Delta y = 0.02$, $\Delta u = 0.1$ and $\Delta v = 0.27$ yielding $u' = u + \Delta u = 1.1$ and $v' = v + \Delta v = -0.73$

16.1 Determine the definiteness of the following symmetric matrices.

$$a)\ \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad b)\ \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \quad c)\ \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix} \quad d)\ \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$e)\ \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \quad f)\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad g)\ \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

Answer: In each case, we start by computing the leading principal minors. If necessary, we will compute all principal minors.

a) **Positive Definite** because $H_1 = 2 > 0$ and $H_2 = 1 > 0$.

b) **Indefinite** because $H_1 = -3 < 0$ and $H_2 = -1 < 0$.

c) **Negative Definite** because $H_1 = -3 < 0$ and $H_2 = 2 > 0$.

d) **Positive Semidefinite** because $H_1 = 2 > 0$, $H_2 = 0$, and $a_{22} = 8 > 0$.

e) **Indefinite** because $H_1 = 1 > 0$, $H_2 = 0$, $H_3 = -25 < 0$.

f) **Negative Semidefinite** because $H_1 = -1 < 0$, $H_2 = 0$, $H_3 = 0$, $a_{22} = -4 < 0$ $a_{33} = -2 < 0$, and the other two second order minors are both $2 > 0$.

g) **Indefinite** because $H_1 = 1 > 0$, $H_2 = 2 > 0$, $H_3 = -10 < 0$ and $H_4 = 65 > 0$.

16.6 Determine the definiteness of the following constrained quadratics.

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

c) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

d) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

e) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 - x_3 = 0$.

Answer:

a) Here $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

There are $n = 2$ variables and $m = 1$ constraints, so we must look at the last leading principal minor, H_3. We find $H_3 = 2$. It has the same sign as $(-1)^n = 1$, so the constrained quadratic is **negative definite**.

b) The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Again, $n = 2$ and $m = 1$, so we must look at the last leading principal minor. Then $H_3 = -1$ which has the same sign as $(-1)^m = -1$, so it is **positive definite**.

c) The bordered Hessian is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{pmatrix}.$$

Now there are $n = 3$ variables and $m = 2$ constraints, so we look at the last leading principal minor. Then $H_5 = 16$ which has the same sign as $(-1)^m = -1$, so it is **positive definite**.

d) The bordered Hessian is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{pmatrix}.$$

As in (c), there are $n = 3$ variables and $m = 2$ constraints. We again check the last leading principal minor. We find $H_5 = 16$, which has the same sign as $(-1)^m = -1$. It is **positive definite**.

e) The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & -3 \\ -1 & 0 & -3 & -1 \end{pmatrix}.$$

There are $n = 3$ variables and $m = 1$ constraints, so we check the last 2 leading principal minors. We obtain $H_3 = 3$ and $H_4 = 4$. Both have the same sign, but the sign of H_4 is neither $(-1)^m = -1$ or $(-1)^n = -1$, and the form is **indefinite** on the constraint set.