Homework Assignment #6

16.2 Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form on $\mathbb{R}^n$. By evaluating Q on each of the coordinate axes in $\mathbb{R}^n$, prove that a necessary condition for a symmetric matrix to be positive definite (positive semidefinite) is that all the diagonal entries be positive (nonnegative). State and prove the corresponding result for negative and negative semidefinite matrices. Give an example to show that this necessary condition is not sufficient.

Answer: The corresponding result is the following theorem.

THEOREM 1. *A necessary condition for for a symmetric matrix to be negative definite (negative semidefinite) is that all the diagonal entries are negative (nonpositive).*

PROOF. Calculate $Q(\mathbf{e}^i) = a_{ii}$. This must be negative for all i if Q is negative definite, nonpositive for all i if Q is negative semidefinite.

To show the necessary conditions are not sufficient, consider

$$A_1 = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}.$$

Then A_1 satisifes the necessary conditions for negative definiteness, and A_2 satisfies the necessary conditions for negative semidefiniteness. But the corresponding quadratic forms take the values $Q_1(1,1) = 6$ and $Q_2(1,1) = 8$, showing that A_1 is not negative definite and that A_2 is not negative semidefinite.

16.3 Using the method of the previous exercise, sketch a proof that if A is positive (or negative) definite, then every principal submatrix of A is also positive (or negative) definite.

Answer: Suppose A is positive definite and A_k is a principal submatrix of order k. Let $\mathbf{x} \in \mathbb{R}^K$. Define $\mathbf{y}$ by $y_i = 0$ for rows i that are omitted in A_k and succesively fill in the values of $\mathbf{x}$ for the other y_i. Then $\mathbf{x}^\mathbf{T} A_k\mathbf{x} = \mathbf{y}^\mathbf{T} A\mathbf{y}$. Since A is positive, A_k is also positive. If A is definite, $\mathbf{x}^\mathbf{T} A_k\mathbf{x} = 0$ implies $\mathbf{y}^\mathbf{T} A\mathbf{y} = 0$, so $\mathbf{y} = \mathbf{0}$. It then follows that $\mathbf{x} = \mathbf{0}$. The negative case is similar.

17.2 For each of the following functions defined on $\mathbb{R}^3$, find the critical points and classify them as local max, local min, saddle point or "can't tell":

$$a)\ x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z,$$
$$b)\ (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$$

Answer:

a) The first-order conditions are:

$$0 = 2x + 6y - 10$$
$$0 = 6x + 2y - 3z - 5$$
$$0 = -3y + 8z - 21$$

and the system has solution $(x, y, z) = (2, 1, 3)$ by Cramer's Rule. The Hessian is

$$H = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}.$$

The first two leading principal minors of the Hessian are $H_1 = 2 > 0$ and $H_2 = -32 < 0$. This is indefinite, so all critical points are saddlepoints.

b) To simplify notation, set $\psi = e^{-(x^2+y^2+z^2)}$. The first-order conditions are:

$$\begin{aligned}
0 &= 2x[1 - x^2 - 2y^2 - 3z^2]\psi \\
0 &= 2y[2 - x^2 - 2y^2 - 3z^2]\psi \\
0 &= 2z[3 - x^2 - 2y^2 - 3z^2]\psi.
\end{aligned}$$

At most one of x, y, and z can be non-zero, otherwise the first-order conditions would give two different values for $x^2 + 2y^2 + 3z^2$. The critical points are $(0, 0, 1)$, $(0, 0, -1)$, $(0, 1, 0)$, $(0, -1, 0)$, $(1, 0, 0)$, $(-1, 0, 0)$, and $(0, 0, 0)$.

We now consider the Hessian. The second derivatives of our function (which we will call f) are:

$$\begin{aligned}
f_{xx} &= [2(1 - 2x^2)(1 - x^2 - 2y^2 - 3z^2) - 4x^2]\psi \\
f_{yy} &= [2(1 - 2y^2)(2 - x^2 - 2y^2 - 3z^2) - 8x^2]\psi \\
f_{zz} &= [2(1 - 2z^2)(3 - x^2 - 2y^2 - 3z^2) - 12x^2]\psi \\
f_{xy} &= [-8xy - 4xy(1 - x^2 - 2y^2 - 3z^2)]\psi \\
f_{xy} &= [-12xz - 4xz(1 - x^2 - 2y^2 - 3z^2)]\psi \\
f_{xy} &= [-12xy - 4yz(2 - x^2 - 2y^2 - 3z^2)]\psi.
\end{aligned}$$

Note that the cross partials are zero at every critical point. Moreover, only the squared values of the variables appear. We therefore must consider three Hessians.

$$\begin{aligned}
H(\pm 1, 0, 0) &= \frac{1}{e}\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\
H(0, \pm 1, 0) &= \frac{1}{e}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
H(0, 0, \pm 1) &= \frac{1}{e}\begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \end{pmatrix} \\
H(0, 0, 0) &= \frac{1}{e}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.
\end{aligned}$$

The first two pairs are indefinite, so the critical points $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$ are saddlepoints. The last pair is negative definite, so $(0, 0, \pm 1)$ are local (in fact, global) maxima. Finally, $H(0, 0, 0)$ is positive definite, so $(0, 0, 0)$ is a local minimum.

17.4 A firm uses two inputs to produce a single product. If its production function is $Q = x^{1/4}y^{1/4}$ and if it sells its output for a dollar a unit and buys each input for \$4 dollars a unit, find its profit-maximizing input bundle. (Check the second order conditions.)

Answer: Under these conditions, profit is $\pi(x, y) = x^{1/4}y^{1/4} - 4x - 4y$. We treat this as a unconstrained profit maximization problem since neither $x = 0$ nor $y = 0$ can yield positive profit. The first order conditions are $(1/4)x^{-3/4}y^{1/4} = 4$ and $(1/4)x^{1/4}y^{-3/4} = 4$. Dividing the first by the second yields $y/x = 1$, so $x = y$. Then substitute back in the production function to find $(1/4)x^{-1/2} = 4$. The solution is $x = y = 4^{-4} = 1/256$, which yields output $Q = 1/4$.

The Hessian of the objective function is:

$$\mathbf{H} = \frac{1}{16}\begin{pmatrix} -3x^{-7/4}y^{1/4} & x^{-3/4}y^{1/4} \\ x^{-3/4}y^{1/4} & -3x^{1/4}y^{-7/4} \end{pmatrix}.$$

At $x = y = 1/256$, this becomes

$$\mathbf{H} = \frac{1}{16}\begin{pmatrix} -3x^{-3/2} & x^{-1/2} \\ x^{-1/2} & -3x^{-3/2} \end{pmatrix}.$$

The leading principal minors are $\mathbf{H}_1 = -(3/16)x^{-3/2} < 0$ and $\mathbf{H}_2 = (1/16x)(9x^{-2} - 1) > 0$, indicating that $\mathbf{H}$ is negative definite. This means $(x, y) = (1/256, 1/256)$ is a maximum.

17.6 Dingbat Airlines has regular flights between Ypsilanti and Kalamazoo. It can treat business and pleasure travelers as separate markets by demanding advance purchase and Saturday night stay-over for pleasure travelers. Supose that it notes a demand function of $Q = 16 - p$ for business travelers and a demand function $Q = 10 - p$ for pleasure travelers and that it has a cost function for all travelers of $C(Q) = 10 + Q^2$. How much should it charge in each market to maximize its profit?

Answer: We solve this in terms of prices. It can also be solved in terms of quantities.

Let p_b and p_p be the prices for business and pleasure travelers, respectively. Treating them as separate markets, revenue is $p_b(16 - p_b) + p_p(10 - p_p)$, the quantity produced is $26 - p_b - p_b$ and cost is $10 + (26 - p_b - p_p)^2$. Profit is $\pi(p_b, p_p) = p_b(16 - p_b) + p_p(10 - p_p) - 10 - (26 - p_b - p_p)^2$. The first order conditions are

$$\frac{\partial \pi}{\partial p_b} = 68 - 4p_b - 2p_p = 0 \text{ and}$$
$$\frac{\partial \pi}{\partial p_p} = 62 - 2p_b - 4p_p = 0.$$

The Hessian is:

$$\mathbf{H} = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}.$$

The leading principal minors are $\mathbf{H}_1 = -4$ and $\mathbf{H}_2 = 12$, indicating that we have a maximium. The profit maximizing prices are $p_b = 37/3$ and $p_p = 28/3$ with corresponding quantities $q_b = 11/3$ and $q_p = 2/3$.