Homework Assignment #7

18.3 Find the point on the parabola $y = x^2$ that is closest to the point $(2, 1)$. (Estimate the solution of the cubic equation that results.)

Answer: We will minimize the squared distance instead. Although one could use a Lagrangian, direct substitution is easier here. We substitute $y = x^2$ in the squared distance $(x-2)^2 + (y-1)^2$. This gives us $f(x) = (x-2)^2 + (x^2-1)^2$.

We then compute $f'(x) = 2(x-2) + 2(x^2-1)2x = 4x^3 - 2x - 4 = 0$. We divide by two to obtain $F(x) = 2x^3 - x - 2 = 0$.

Note $F'(x) = 6x - 1$. We try the solution $x = 1$, but $f(x) = -1$. To raise F by one, we try $\Delta x = \Delta F/F'(x)$. At $x = 1$, we find $\Delta x = 1/5 = .2$. Now $f(1.2) = 0.256$. As $F'(1.2) = 7.2$, the new $\Delta x = -.256/7.2 = -0.036$. The new $x = 1.164$, where $f(x) = -0.01$ and $F(x) = 5.984$. The new $\Delta x = 0.01/5.984 = .00167$. At this point 1.165 is a little low and 1.166 is a little high, so we take $x \approx 1.165$, so $y \approx 1.357$. Note that the second derivative is positive, so it is a local minimum.

18.17 Minimize $x^2 - 2y$ subject to the constraints $x^2 + y^2 \le 1$, $x \ge 0$, $y \ge 0$.

Answer: We maximize $2y - x^2$. This has a solution as the objective is continuous and the constraint set is compact. The Lagrangian is $\mathcal{L} = 2y - x^2 - \lambda(x^2 + y^2 - 1) + \mu_x + \mu_y$. The first-order conditions become $-2x - 2x\lambda + \mu_x = 0$ and $2 - 2\lambda y + \mu_y = 0$. We rewrite these as $2x(1+\lambda) = \mu_x$ and $2\lambda y = 2 + \mu_y$. The second equation implies $\lambda > 0$ and $y > 0$, so $\mu_y = 0$ and $x^2 + y^2 = 1$ by complementary slackness.

This makes the first order conditions $2x(1+\lambda) = \mu_x$ and $2\lambda y = 1$. Note that $x > 0$ implies $\mu_x > 0$, which contradicts complementary slackness. Therefore $x = 0$. But then $x^2 + y^2 = 1$ and $y \ge 0$, so $y = 1$. The only solution is $(x, y) = (0, 1)$, $\lambda = 1/2$, $\mu_x = \mu_y = 0$.

The function takes the value -2 at the minimum.

19.18 Consider the problem of maximizing $x_1^2 x_2$ on the constraint set $2x_1^2 + x_2^2 = a$, as in Example 18.5. Use the Implicit Function Theorem directly on the first order conditions of this problem to prove that the solutions $x_1(a)$, $x_2(a)$, $\lambda(a)$ depend smoothly on the parameter a near $a = 3$.

Answer: The Lagrangian is $\mathcal{L} = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - a)$. Note that $x_1^2 x_2 > 0$ is possible, so $x_1 \ne 0$ and $x_2 \ne 0$ at the maximum. This allows us to divide by x_1 or x_2. The first-order conditions and constraint can be simplified to:

$$\begin{aligned} 0 &= x_2 - 2\lambda \\ 0 &= x_1^2 - 2\lambda x_2 \\ 0 &= 2x_1^2 + x_2^2 - a. \end{aligned}$$

We have three equations and three unknowns (x_1, x_2, λ). In order to apply the Implicit Function Theorem, the derivative of this system with respect to (x_1, x_2, λ) must be non-singular. The derivative is:

$$\begin{pmatrix} 0 & 1 & -2 \\ 2x_1 & -2\lambda & -x_2 \\ 4x_1 & 4x_2 & 0 \end{pmatrix}$$

This has determinant $-4x_1(5x_2 + 4\lambda)$. When (x_1, x_2, λ) solve the equations, $2\lambda = x_2$ and the determinant reduces to $-40x_1x_2$, which is not zero. The Implicit Function Theorem then tells us that $(x_1(a), x_2(a), \lambda(a))$ are C^1 functions for any $a > 0$, in particular for a near 3.

20.1 Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?

$$a)\ 3x^5y + 2x^2y^4 - 3x^3y^3, \qquad b)\ 3x^5 + 2x^2y^4 - 3x^3y^4,$$
$$c)\ x^{1/2}y^{-1/2} + 3xy^{-1} + 7, \qquad d)\ x^{3/4}y^{1/4} + 6x,$$
$$e)\ x^{3/4} + 6x + 4, \qquad f)\ \frac{(x^2 - y^2)}{(x^2 + y^2)} + 3.$$

Answer: Function (a) is homogeneous of degree 6. Function (b) is not homogeneous as the first 2 terms are h.d. 6 and the third is h.d. 7. Function (c) is homogeneous of degree 0. Function (d) is homogeneous of degree 1. Function (e) is not homogeneous due to the constant term. Function (f) is homogeneous of degree 0.

20.17 Which of the following functions are homothetic? Give a reason for each answer.

$$a)\ e^{x^2y}e^{xy^2}, \qquad b)\ 2\log x + 3\log y, \qquad c)\ x^3y^6 + 3x^2y^4 + 6xy^2 + 9,$$
$$d)\ x^2y + xy, \qquad e)\ x^2y^2/(xy + 1).$$

Answer: Both (a) and (b) are increasing transformations of a homogeneous function: (a) $e^{x^2y}e^{xy^2} = e^{x^2y+xy^2}$ where $x^2y + xy^2$ is homogeneous of degree 3 and (b) $\log(x^2y^3)$ where x^2y^3 is homogeneous of degree 5.

Case (c) is trickier. Here $f(x, y) = g(xy^2)$ where $z = xy^2$ is homogeneous of degee 3 and $g(z) = z^3 + 3z^2 + 6z + 9$. Note that $g' = 3(z + 1)^2 + 3 > 0$, so g is an increasing function. As an increasing transformation of a homogeneous function, (c) is homothetic.

Case (d) is not homothetic. This can be seen by comparing the values at some $\mathbf{x}_1$ and $\mathbf{x}_2$, and then $\lambda\mathbf{x}_1$ and λ_2. We have $f(1, 5) = 10 < f(3, 1) = 12$, but $f(0.4, 2) = 1.12 > 1.056 = f(1.2, 0.4)$ (here $\lambda = 0.4$).

For case (e) note that $f(x, y) = g(xy)$ where $g(z) = z^2/(z + 1)$. As in case (c), we must show that g is increasing. Now $g' = (z^2 + 2)/(z + 1)^2 > 0$, so (e) is homothetic as an increasing transformation of a homogeneous function.