Mathematical Economics Exam #1, September 25, 2018

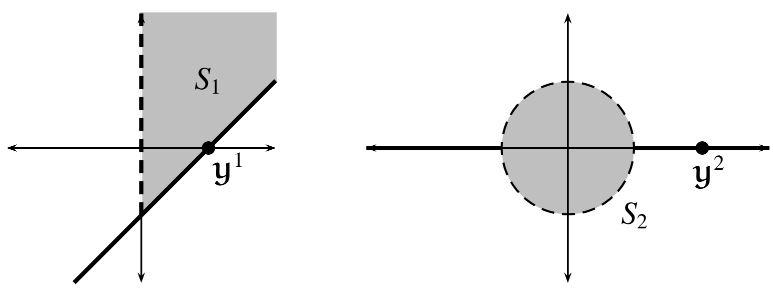
1. Consider the following sets in $\mathbb{R}^2$.

   a) $S_1 = \{\mathbf{x} \in \mathbb{R}^2 : x_1 - x_2 \le 1, x_1 > 0\}$.

   b) $S_2 = B_1(\mathbf{0}) \cup \{\mathbf{x} \in \mathbb{R} : x_2 = 0\}$.

   Identify which of the sets are open or closed (if any), and justify your answer. You may find it helpful to sketch the set.

   Answer: We start with a sketch of the sets:



   Both sets are neither open nor closed. We show this be considering boundary points that either are not in the closure of S_i or not in the interior.

   For S_1, consider the sequence $\mathbf{x}^n = (1/n, -1+1/n) \in S_1$. This has limit $\mathbf{x} = (0,-1) \notin S_1$ as $x_1 = 0$, so S_1 is not closed. Then consider the point $\mathbf{y}^1 = (1,0) \in S_1$. Given $\varepsilon > 0$, the point $\mathbf{x} = (1, -\varepsilon/2) \notin S_1$ and $\mathbf{x} \in B_\varepsilon(\mathbf{y}^1)$. This shows that every ball around $\mathbf{y}^1 \in S_1$ contains points not in S_1, so S_1 is not open.

   We now consider S_2. Here the sequence $\mathbf{x}^n = (0, 1-1/n) \in B_1(\mathbf{0}) \subset S_2$ and $\mathbf{x}^n \to (0,1) \notin S_2$, so S_2 is not closed. To show S_2 is not open, consider the point $\mathbf{y}^2 = (2,0) \in S_2$. Then $(10, \varepsilon/2) \in B_\varepsilon(\mathbf{y}^2)$. But $(2, \varepsilon/2) \notin S_2$, so any ball about $(2,0)$ contains points that are not in S_2. This means that S_2 is not open.

2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   a) Compute the rank of $\mathbf{A}$.

   b) How many vectors $\mathbf{x}$ obey $\mathbf{A}\mathbf{x} = \mathbf{0}$?

   c) Does $\mathbf{A}^{-1}$ exist? If so, find it.

   Answer:

   a) Here $\det \mathbf{A} = +1$, showing that $\mathbf{A}$ is invertible. Since $\mathbf{A}$ is a 2×2 matrix, it follows that $\operatorname{rank} \mathbf{A} = 2$

b) Since A is invertible, the homogeneous linear system has a unique solution, $\mathbf{x} = \mathbf{0}$.

c) We found in part (a) that $\mathbf{A}$ has an inverse. It is easily computed and is $\mathbf{A}^{-1} = -\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

3. Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 4 \\ 0 & -2 & -4 & -4 \\ 3 & 1 & -1 & 5 \end{pmatrix}$$

and let $V = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$.

a) Show that V is a vector subspace of $\mathbb{R}^4$.

b) Find the dimension of V.

c) Find a basis for V.

Answer:

a) Suppose $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$. Then $\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} = 0 + 0 = 0$, so $\alpha\mathbf{x} + \beta\mathbf{y} \in V$. This shows that V is a vector subspace of $\mathbb{R}^3$.

b) We row-reduce $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 4 \\ 0 & -2 & -4 & -4 \\ 3 & 1 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & -4 & -4 \\ 0 & 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This has rank two, so the dimension of V is $\dim V = 4 - \operatorname{rank} A = 2$ by the Fundamental Theorem of Linear Algebra.

c) Using the row-reduced form of $\mathbf{A}$, we find that

$$x_1 = x_3 - x_4$$
$$x_2 = -2x_3 - 2x_4$$

One way to find a basis is to set $x_3 = 1, x_4 = 0$ and $x_3 = 0$, $x_4 = 1$. This yields $\mathbf{x}_1 = (1, -2, 1, 0)^{\mathsf{T}}$, $\mathbf{x}_2 = (-1, -2, 0, 1)^{\mathsf{T}}$. By construction, $\mathbf{x}_1, \mathbf{x}_2 \in V$. These are linearly independent, and so must span V. In fact, if $\mathbf{x} \in V$, $\mathbf{x} = x_3\mathbf{x}_1 + x_4\mathbf{x}_2$.

4. Consider $\mathbb{R}^N$ with a inner product $x \cdot y$. As usual, define the associated norm by $\|x\| = \sqrt{x \cdot x}$. Show that for all $x, y \in \mathbb{R}^N$,

$$x \cdot y = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right].$$

Answer:

$$\begin{aligned} \|x+y\|^2 - \|x-y\|^2 &= \|x\|^2 + x\cdot y + y\cdot x + \|y\|^2 - \|x\|^2 + x\cdot y + y\cdot x - \|y\|^2 \\ &= 2x\cdot y + 2y\cdot x \\ &= 4x\cdot y. \end{aligned}$$

The result then follows.