Mathematical Economics Exam #2, Oct. 30, 2018

1. Consider the quadratic form $Q(\mathbf{x}) = x_1^2+2x_1x_2+x_2^2+2x_2x_3+x_3^2$. Does Q have a maximum, minimum, or neither at $\mathbf{x} = \mathbf{0}$ under the constraint $x_2 + x_3 = 0$.

   Answer: To answer this we form the bordered Hessian.

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

   Since there are $n = 3$ variables and $m = 1$ constraint, we must check the last two leading principal minors. They are $\mathbf{B}_3 = -1 < 0$ and $\mathbf{B}_4 = +1 > 0$.

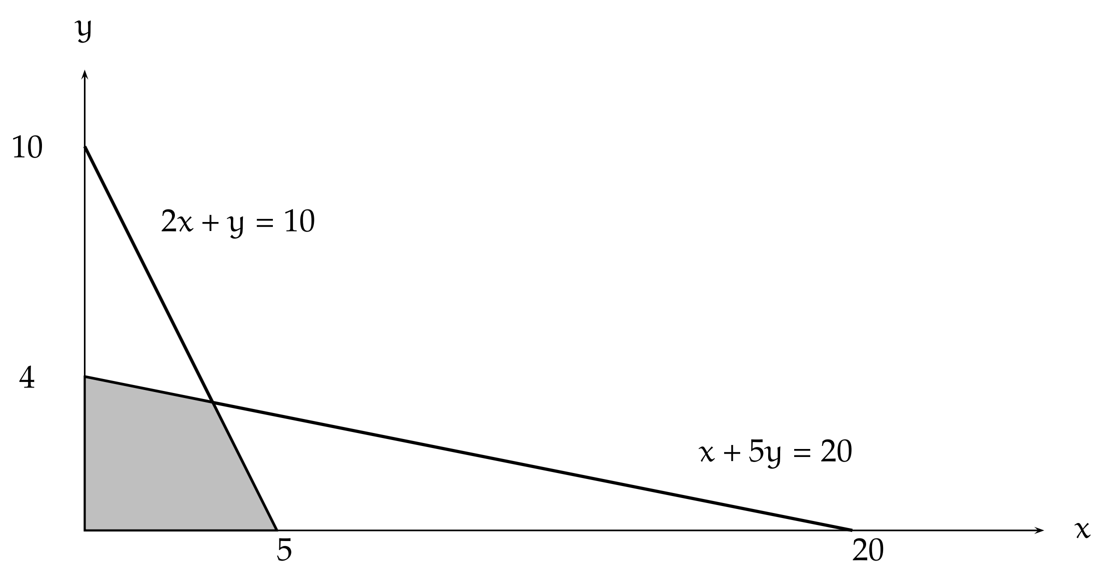
   Now $\mathbf{B}_4 > 0$ while $(-1)^n = -1 < 0$, so it fails the negative definite test. Also, $\mathbf{B}_3$ and $\mathbf{B}_4$ have opposite signs, so if fails the positive definite test. It follows that $\mathbf{x} = \mathbf{0}$ is neither a maximum nor minimum point.

   Another way to see that is to substitute the constraint into the quadratic form, leaving $x_1^2 + 2x_1x_2$, which is positive at $\varepsilon(1, 1, -1)$ and negative at $\varepsilon(1, -1, 1)$.

2. Consider the problem of maximizing $x+y$ under the constraints $2x+y \le 10$, $x+5y \le 20$, $x \ge 0$, and $y \ge 0$.

   a) Sketch the feasible set.

   Answer: See following diagram.



b) Is constraint qualification satisfied?

Answer: Two of the constraints must be rewritten as $-x \leq 0$ and $-y \leq 0$. Then

$$dg = \begin{pmatrix} 2 & 1 \\ 1 & 5 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Each line is non-zero, so if only one constraint binds, constraint qualification is satisfied. Part (a) makes it clear that at most two constraints can bind. Any pair of rows of dg are linearly independent, so the rank is two in any of these cases. This means that the non-degenerate constraint qualification condition (NDCQ) is satisfied.

c) Find all solutions to the maximization problem.

Answer: The Lagrangian is $\mathcal{L} = x+y+\lambda_1 x+\lambda_2 y+\lambda_3(10-2x-y)+\lambda_4(20-x-5y)$. The first order conditions are

$$\begin{aligned} 0 &= 1 + \lambda_1 - 2\lambda_3 - \lambda_4 \\ 0 &= 1 + \lambda_2 - \lambda_3 - 5\lambda_4. \end{aligned}$$

This conditions imply that at least one of λ_3, λ_4 is positive. By complementary slackness at least one of constraints 3 and 4 must bind.

We now rewrite the first order conditions

$$\begin{aligned} 1 + \lambda_1 &= 2\lambda_3 + \lambda_4 \\ 1 + \lambda_2 &= \lambda_3 + 5\lambda_4. \end{aligned}$$

We consider the case $x = 0$. Since either $2x + y = 10$ or $x + 5y = 20$, we find either $y = 10$ or $y = 4$. Notice that $y = 10$ violates $x + 5y \leq 20$, so we conclude $y = 4$. By complementary slackness, $\lambda_2 = 0$ and $\lambda_3 = 0$. The first-order conditions are now $1 + \lambda_1 = \lambda_4$ and $1 = 5\lambda_4$, which implies $\lambda_1 = -4/5 < 0$. This is impossible, so there is no solution here.

Suppose instead $y = 0$. Then either $x = 5$ or $x = 20$. Now $x = 20$ violates the constraint that $2x + y \leq 10$, so we must have $x = 5$. Then $\lambda_1 = \lambda_4 = 0$ by complementary slackness. The first-order conditions become $1 = 2\lambda_3$ and

$1 + \lambda_2 = \lambda_3$. This requires $\lambda_3 = 1/2$ and $\lambda_2 = -1/2$, which is impossible. Again, there is no solution here.

Finally suppose $x, y > 0$. Then $\lambda_1 = \lambda_2 = 0$. It follows that

$$1 = 2\lambda_3 + \lambda_4$$
$$1 = \lambda_3 + 5\lambda_4.$$

This has solution $\lambda_3 = 4/9$, $\lambda_4 = 1/9$, implying that both $2x + y = 10$ and $x + 5y = 20$. The solution to this system is $x = y = 10/3$, with objective equal to $20/3$.

3. Consider the function $f(x, y) = x^{1/3}+y^{1/4}$ defined on the set $F = \{(x, y) \in \mathbb{R}^2 : x^2+15y^2 \leq 20\}$. **Note:** I overlooked the fact that negative values are a problem here. I meant $f(x, y) = |x|^{1/3} + |y|^{1/4}$.

   a) Show that F is a compact set.

   b) Show that f has both a maximum and minimum value on F.

   Answer:

   a) The function $g(x, y) = x^2 + 15y^2$ is continuous because it is a polynomial. That means that $F = g^{-1}(-\infty, 20]$ is closed as the inverse image of a closed set under a continuous function. Moreover, if $(x, y) \in F$, then $|x| \leq \sqrt{20}$ and $|y| \leq 2/\sqrt{3}$, showing that F is bounded. As a closed and bounded set, F is compact.

   b) Now f is a continuous function. By the Weierstrass Theorem, it has both a maximum and minimum over the compact set F.

4. Let $f(x, y, z) = (x^2 + 4y^2)(z + 1)^2$.

   a) If $(x, y) = (1, 1/2)$ and $f(x, y, z) = 18$, what values may z take.

   b) When $(x, y) = (1, 1/2)$, can z be written as a C^1 function of (x, y) near $(1, 1/2)$? Explain.

   c) If you answered (b) affirmatively, call such a function g and find $dg(1, 1/2)$.

   Answer:

   a) Set $18 = f(1, 1/2, z) = 2(z + 1)^2/2$. Then solve to find $z = 2$ or $z = -4$..

   b) We calculate $\partial f/\partial z = 2(z + 1)(x^2 + 4y^2)$. Using $(x, y, z) = (1, 1/2, 2)$ and $(x, y, z) = (1, 1/2, -4)$, we find $\partial f/\partial z = \pm 12$. As this is non-zero, we can apply the Implicit Function Theorem about either solution to write z as a function of (x, y).

   c) The Implicit Function Theorem tells us that $dg(1, 1/2) = (\partial f/\partial z)^{-1} d_{(x,y)}f$. We already found $\partial f/\partial z = \pm 12$. Now $d_{(x,y)}f(1, 1/2, -1 \pm 3) = (18, 36)$ so $df(1, 1/2) =$

$\pm(1.5, 3)$ depending on which value of z we picked.