Mathematical Economics Final, December 10, 2019

1. On $\mathbb{R}^2$, maximize $3x + y$ under the constraint $x^2 + y^2 \le 10$. Note that there are no non-negativity constraints. Don't forget to check constraint qualification. What do the second order conditions tell you? Is your solution a maximum?

   Answer: The derivative of the constraint is $(2x, 2y)$, which has rank one except at $(0, 0)$. The maximum is not at $(0, 0)$ because $(1, 1)$ is feasible and better, so constraint qualification is satisfied.

   The Lagrangian is $\mathcal{L} = 3x + y - \lambda(x^2 + y^2 - 10)$. The first order conditions are

$$
\begin{aligned}
0 &= 3 - 2\lambda x \\
0 &= 1 - 2\lambda y.
\end{aligned}
$$

   To satisfy these equations, x, y, and λ must all be non-zero (again insuring constraint qualification is satisfied). Combining the first-order conditions, we find $x = 3y$. Complementary slackness implies that $x^2 + y^2 = 10$ because λ is non-zero. It follows that $10y^2 = 10$. Then $y = \pm 1$ and $x = \pm 3$.

   Our two solutions are $(x, y) = \pm(1, 3)$. The objective takes the values ± 6. The only possible solution is $(1, 3)$. In this case $\lambda = 1/2$.

   That leaves the second order conditions. In this case, the Hessian of the Lagrangian is

$$
\mathbf{H} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},
$$

   which is negative definite.

   Alternatively, one can form the bordered Hessian using the Lagrangian and evaluating at $(1/2, 3/2)$ to obtain

$$
\mathbf{B} = \begin{pmatrix} 0 & 3 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.
$$

   The determinant is $10 > 0$. Since there are two variables, this must have the same sign as $(-1)^2 = +1$, which it does. The second order conditions are satisfied.

2. Let $u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$. Solve the expenditure minimization problem for $\bar{u} \ge 0$ and

$\mathbf{p} \gg \mathbf{0}$. Identity the Hicksian demands and find the expenditure function e.

$$\begin{aligned} e(\mathbf{p}, \bar{u}) = \min\ & \mathbf{p}\cdot\mathbf{x} \\ \text{s.t. }\ & u(\mathbf{x}) \geq \bar{u} \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Answer: Taking into account that this is a minimization problem, we form the Lagrangian

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda_0 (x_1^{1/2} + x_2^{1/2} - \bar{u}) - \lambda_1 x_1 - \lambda_2 x_2.$$

There are three constraints. At most two can bind unless $\bar{u} = 0$. In that case, $(x_1, x_2) = (0, 0)$ minimizes expenditure. In the remainder, we assume $\bar{u} > 0$, implying that at least one of the x_i is positive.

The first-order conditions are

$$\begin{aligned} 0 &= p_1 - \lambda_0 \frac{1}{2} x_1^{-1/2} - \lambda_1 \\ 0 &= p_2 - \lambda_0 \frac{1}{2} x_2^{-1/2} - \lambda_2. \end{aligned}$$

If either $x_1 = 0$ or $x_2 = 0$, these equations have no solution. Thus $x_1 > 0$ and $x_2 > 0$. Constraint qualification is then satisfied as $\frac{1}{2}(x_1^{1/2}, x_2^{1/2x})$ is non-zero. Complementary slackness implies $\lambda_1 = \lambda_2 = 0$. The first-order conditions reduce to

$$p_1 = \lambda_0 \frac{1}{2} x_1^{-1/2}, \quad p_2 = \lambda_0 \frac{1}{2} x_2^{-1/2}$$

Since $x_1 > 0$ and $p_1 > 0$, $\lambda_0 > 0$. We now divide to find

$$\frac{p_1}{p_2} = \frac{x_2^{1/2}}{x_1^{1/2}}$$

so

$$x_1^{1/2} = \left(\frac{p_2}{p_1}\right) x_2^{1/2} \tag{1}$$

Because $\lambda_0 > 0$, complementary slackness tells us that $x_1^{1/2} + x_2^{1/2} = \bar{u}$. Substituting Equation (1) in this equality, we obtain

$$\bar{u} = \left(\frac{p_1 + p_2}{p_1}\right) x_2^{1/2}$$

It follows that the Hicksian demands are

$$x_1 = \left(\frac{p_2\bar{u}}{p_1+p_2}\right)^2 \text{ and } x_2 = \left(\frac{p_1\bar{u}}{p_1+p_2}\right)^2$$

Finally, we substitute the Hicksian demands in the objective to find

$$e(\mathbf{p},\bar{u}) = \bar{u}\left(\frac{p_1p_2}{p_1+p_2}\right).$$

Note that these formulas also apply when $\bar{u}=0$.

3. Consider the differential equation $\ddot{y} - 2\dot{y} - 3y = 3t^2$ with initial conditions $y(0) = -22/9$ and $\dot{y}(0) = 4/3$.

   a) Find the general solution of the associated homogeneous equation.

   b) Find a particular solution of the inhomogeneous equation.

   c) Find the solution that obeys the initial conditions.

   Answer:

   a) We substitute $y = e^{rt}$ to find the characteristic equation, $r^2 - 2r - 3 = 0$. This has solutions $r = -1$ and $r = 3$. The corresponding general solution to the homogeneous equation is $c_1e^{-t} + c_2c^{3t}$.

   b) To find a particular solution, we try the form $y = at^2 + bt + c$. Then $\dot{y} = 2at + b$ and $\ddot{y} = 2a$. The particular solution must obey

$$\begin{aligned} 3t^2 &= 2a - 2(2at+b) - 3(at^2+bt+c) \\ &= -3at^2 - (4a+3b)t + (2a-2b-3c) \end{aligned}$$

   We must have $-3a = 3$, $4a+3b = 0$, and $2a-2b-3c = 0$. Then $a = -1$, $b = 4/3$, and $c = 2(a-b)/3 = -14/9$. The particular solution is $-t^2 + 4t/3 - 14/9$.

   c) Combining (a) and (b), we find the general solution to the original equation is $y(t) = c_1e^t + c_2e^{3t} - t^2 + 4t/3 - 14/9$. Then $y(0) = c_1 + c_2 - 14/9 = -22/9$, so $c_1 + c_2 = -8/9$. Also, $\dot{y}(0) = -c_1 + 3c_2 + 4/3 = 4/3$, so $-c_1 + 3c_2 = 0$. Solving for c_1, c_2, we obtain $c_1 = -2/9$ and $c_2 = -6/9$. The solution is

$$y(t) = -\frac{2}{9}\left(3e^{-t} + e^{3t}\right) - t^2 + \frac{4}{3}t - \frac{14}{9}.$$

4. Let $f(x,y) = (xy)^{1/2}$. Is f concave on $\mathbb{R}^2_{++}$?

Answer: We start by computing the partial derivatives, $\partial f/\partial x = (y/x)^{1/2}/2 = f/2x$ and $\partial f/\partial y = (x/y)^{1/2}/2 = f/2y$. Then the second partial derivatives are

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -\frac{y^{1/2}}{3x^{3/2}} = -\frac{f}{4x^2}, \\ \frac{\partial^2 f}{\partial y^2} &= -\frac{x^{1/2}}{4y^{3/2}} = -\frac{f}{4y^2}, \text{ and} \\ \frac{\partial^2 f}{\partial x\,\partial y} &= \frac{1}{4x^{1/2}y^{1/2}} = \frac{f}{4xy}. \end{aligned}$$

This yields Hessian

$$\mathbf{H} = \frac{f}{4xy}\begin{bmatrix} -y/x & 1 \\ 1 & -x/y \end{bmatrix}.$$

Both diagonal elements are strictly negative on $\mathbb{R}^2_{++}$, and the determinant of the entire matrix is zero. Since all principal minors have the appropriate sign, the Hessian is negative semi-definite on $\mathbb{R}^2_{++}$. From that, we may conclude that the function f is concave on $\mathbb{R}^2_{++}$.

5. Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & 1 \\ 4 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find all vectors $\bar{\mathbf{x}}$ that are steady states of this differential system.

Answer: Any steady state $\bar{\mathbf{x}}$ obeys $\mathbf{0} = \mathbf{A}\bar{\mathbf{x}} - \mathbf{b}$, so it must solve $\mathbf{b} = \mathbf{A}\bar{\mathbf{x}}$. We solve this by forming the augmented matrix and row-reducing. Noticing that the second row of the augmented matrix is is the sum of the first and third rows, we start by subtracting the first row from the second rather than strictly following the Gauss-Jordan procedure.

$$\begin{bmatrix} 3 & 3 & 1 & 1 \\ 4 & 4 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{(2)-(1)} \begin{bmatrix} 3 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{(3)-(2)} \begin{bmatrix} 3 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(1)/3} \begin{bmatrix} 1 & 1 & 1/3 & 1/3 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(2)-(1)} \begin{bmatrix} 1 & 1 & 1/3 & 1/3 \\ 0 & 0 & 5/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(3/5)(2)} \begin{bmatrix} 1 & 1 & 1/3 & 1/3 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1)-(1/3)(2)} \begin{bmatrix} 1 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that steady states must have the form

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ 1/5 - x \\ 2/5 \end{bmatrix}.$$

for some $x \in \mathbb{R}$.