Homework Assignment #1

6.2 In Missouri, federal income taxes are deducted from income before calculating state taxes. Write out and solve the system of equations which describes the state and federal taxes and charitable contributions of the firm in Example 1 if it were based in Missouri.

Answer: Note that I've corrected the question. The formulas for after-tax profits and for federal tax are unchanged from Example 1. The difference is in the computation of state tax S. State tax is now calculated based on income of \$100,000, minus charitable contributions C and federal tax F. Since the state income tax rate is 5%, the state tax is $S = .05(100,000 - C - F)$. This yields the following system

$$\begin{aligned} C + 0.1S + 0.1F &= 10,000 \\ 0.05C + S + 0.05F &= 5,000 \\ 0.4C + 0.4S + F &= 40,000. \end{aligned}$$

We form the augmented matrix

$$\begin{pmatrix} 1 & 0.1 & 0.1 & 10000 \\ 0.05 & 1 & 0.05 & 5000 \\ 0.4 & 0.4 & 1 & 40000 \end{pmatrix} \xrightarrow{(2)-0.05(1)} \begin{pmatrix} 1 & 0.1 & 0.1 & 10000 \\ 0 & 0.995 & 0.045 & 4500 \\ 0.4 & 0.4 & 1 & 40000 \end{pmatrix}$$
$$\xrightarrow{(3)-0.4(1)} \begin{pmatrix} 1 & 0.1 & 0.1 & 10000 \\ 0 & 0.995 & 0.045 & 4500 \\ 0 & 0.36 & 0.96 & 36000 \end{pmatrix} \xrightarrow{(3)-(.36/.995)(2)} \begin{pmatrix} 1 & 0.1 & 0.1 & 10000 \\ 0 & 0.995 & 0.045 & 4500 \\ 0 & 0 & 0.94372 & 34372 \end{pmatrix}$$

We can find the answer by completing the row reduction or by substituting from the bottom up. Using the latter and rounding to the nearest dollar, the third equation yields $F = 36,422$. We substitute in the second equation to find $S = 2875$. Then the first equation tells us that $C = 6070$.

6.7 Consider the IS-LM model of Example 4 with no fiscal policty ($G = 0$). Suppose the $M_s = M^o$; that is, the intercept of the LM curve is 0. Suppose the $I^o = 1000$, $s = 0.2$, $h = 1500$, $a = 2000$, and $m = 0.16$. Write out the explicit IS-LM system of equations. Solve them for the equilibrium GNP Y and the interest rate r.

Answer: The IS-LM system of Example 4 is:

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o. \end{aligned}$$

Substituting the values above, we obtain:

$$\begin{aligned} .2Y + 2000r &= 1000 \\ .16Y - 1500r &= 0. \end{aligned}$$

We immediately find that $Y = 9375r$ from the second equation. Substituting in the first equation yields $1875r + 2000r = 1000$. Thus $r = 1000/3875 = .258$ and $Y = 2419$ (approximately).

7.3 Solve the following systems by Gausss-Jordan elimination. Note that the third system requires an equation interchange.

$$a)\ \begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned} \qquad b)\ \begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned} \qquad c)\ \begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$$

Answer: We solve these in matrix form by row-reducing the augmented matrix.

$$a)\ \begin{pmatrix} 3 & 3 & 4 \\ 1 & -1 & 10 \end{pmatrix} \xrightarrow{(1)/3)} \begin{pmatrix} 1 & 1 & 4/3 \\ 1 & -1 & 10 \end{pmatrix} \xrightarrow{(2)-(1)} \begin{pmatrix} 1 & 1 & 4/3 \\ 0 & -2 & 26/3 \end{pmatrix} \xrightarrow{-(2)/2} \begin{pmatrix} 1 & 1 & 4/3 \\ 0 & 1 & -13/3 \end{pmatrix} \xrightarrow{(1)-(2)} \begin{pmatrix} 1 & 0 & 17/3 \\ 0 & 1 & -13/3 \end{pmatrix}.$$

Thus $x = 17/3$ and $y = -13/3$ in part (a).

$$b)\ \begin{pmatrix} 4 & 2 & -3 & 1 \\ 6 & 3 & -5 & 0 \\ 1 & 1 & 2 & 9 \end{pmatrix} \xrightarrow{(1)/4} \begin{pmatrix} 1 & 1/2 & -3/4 & 1/4 \\ 6 & 3 & -5 & 0 \\ 1 & 1 & 2 & 9 \end{pmatrix} \xrightarrow{(2)-6(1),(3)-(1)} \begin{pmatrix} 1 & 1/2 & -3/4 & 1/4 \\ 0 & 0 & -1/2 & -3/2 \\ 0 & 1/2 & 11/4 & 35/4 \end{pmatrix} \xrightarrow{(2)\leftrightarrow(1)} \begin{pmatrix} 1 & 1/2 & -3/4 & 1/4 \\ 0 & 1/2 & 11/4 & 35/4 \\ 0 & 0 & -1/2 & -3/2 \end{pmatrix} \xrightarrow{2(2),-12(3)} \begin{pmatrix} 1 & 1/2 & -3/4 & 1/4 \\ 0 & 1 & 11/2 & 35/2 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Thus $z = 3$, $y = 1$, and $x = 2$ in part (b).

$$c)\ \begin{pmatrix} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & -4 & 3 & 0 \end{pmatrix} \xrightarrow{(1)/2} \begin{pmatrix} 1 & 1 & -1/2 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & -4 & 3 & 0 \end{pmatrix} \xrightarrow{(2)-(1),(3)-2(1)} \begin{pmatrix} 1 & 1 & -1/2 & 1 \\ 0 & 0 & 3/2 & -3 \\ 0 & -6 & 4 & -2 \end{pmatrix} \xrightarrow{(2)\leftrightarrow(3)} \begin{pmatrix} 1 & 1 & -1/2 & 1 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & 3/2 & -3 \end{pmatrix} \xrightarrow{-(2)/6,2(3)/2} \begin{pmatrix} 1 & 1 & -1/2 & 1 \\ 0 & 1 & -2/3 & 1/3 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Thus $z = -2$, $y = -1$, and $x = 1$.

7.8 Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

What happens and why?

Answer: We form the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix}.$$

If $a_{11} \neq 0$, row reduction yields

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_1 1} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & b_2 - \frac{a_{21}b_1}{a_1 1} \end{pmatrix}.$$

Setting $\Delta = a_{11}a_{22} - a_{12}a_{21}$, further row reduction yields

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_11} \\ 0 & 1 & \frac{a_{11}b_2 - a_{21}b_1}{\Delta} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & \frac{a_{22}b_1 - a_{12}b_2}{\Delta} \\ 0 & 1 & \frac{a_{11}b_2 - a_{21}b_1}{\Delta} \end{pmatrix}.$$

In fact, if $a_{11} = 0$ and $a_{21} \neq 0$, row reduction gives the same result. If both $a_{11} = a_{21} = 0$, $\Delta = 0$, and this formula no longer applies. In the latter case, there are infinitely many solutions if either $a_{12} \neq 0$ and $b_2 = a_{22}b_1/a_{12}$ or if $a_{22} \neq 0$ and $b_1 = a_{12}b_2/a_{22}$, or if both $a_{12} = a_{22} = 0$. The first two cases have one free variable, the last has two free variables.

7.17

a) Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

Answer: This equation can be reduced to:

$$\begin{aligned} x + y + z &= 13 \\ 4y + 9z &= 48. \end{aligned}$$

It is easily seen that $(x, y, z) = (1, 12, 0)$ and $(x, y, z) = (6, 3, 4)$ are the only solutions in positive integers. (Note that z must be divisible by 4, and so either 0 or 4.)

b) Suppose ou hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

Answer: This can be described by the system

$$\begin{aligned} x + y + z &= 16 \\ x + 5y + 10z &= 94 \end{aligned}$$

where x is the number of pennies, y the number of nickels, and z the number of dimes. It reduces to

$$\begin{aligned} x + y + z &= 16 \\ 4y + 9z &= 78 \end{aligned}$$

The only solution in positive integers is $(x, y, z) = (4, 6, 6)$. Thus you receive 4 pennies, 6 nickels, and 6 dimes.

7.28

a) In IS-LM model (23), use Gaussian elimination to find a general formula involving s, a, m, and h which, when satisfied, will guarantee that system (23) determines a unique value of Y and r in terms of all the other variables.

b) In this case, find an explicit formula for Y and r in terms of all the other variables.

c) Note how changes in each of the exogenous variables affect the values of Y and r.

Answer: The system is:

$$\begin{aligned} sY + ar &= I^* + G \\ mY - hr &= M_s - M^*, \end{aligned} \tag{23}$$

Where all of the parameters s, a, m, h are assumed positive. Let $A = I^* + G$ and $B = M_s - M^*$. Then form the augmented matrix

$$\begin{pmatrix} s & a & A \\ m & -h & B \end{pmatrix}.$$

Since $s > 0$, we may row reduce as follows:

$$\begin{aligned} \begin{pmatrix} s & a & A \\ m & -h & B \end{pmatrix} &\xrightarrow{(2)-(m/s)(1)} \begin{pmatrix} s & a & A \\ 0 & -(sh+am)/s & (sB-mA)/s \end{pmatrix} \\ &\xrightarrow{-(sh+am)/s\times(2)} \begin{pmatrix} s & a & A \\ 0 & 1 & (mA-sB)/(sh+am) \end{pmatrix} \\ &\xrightarrow{(1)-a(2)} \begin{pmatrix} s & 0 & s(hA+aB)/(sh+am) \\ 0 & 1 & (mA-sB)/(sh+am) \end{pmatrix} \end{aligned}$$

a) Whenever $sh + am \neq 0$, the system will have a unique solution (Y, r). Note that positive parameter values imply $sh + am > 0$.

b) Using $A = I^* + G$ and $B = M_s - M^*$, the solution is then

$$Y = \frac{h(I^* + G) + a(M_s - M^*)}{sh + am}, \quad r = \frac{m(I^* + G) - s(M_s - M^*)}{sh + am}.$$

c) Now $\partial Y/\partial I^* = \partial Y/\partial G = sh/(sh+am)$, $\partial Y/\partial M_s = -\partial Y/\partial M^* = sa/(sh+am)$ and $\partial r/\partial I^* = \partial Y/\partial G = m/(sh+am)$, $\partial r/\partial M_s = -s/(sh+am) = -\partial r/\partial M^*$.